Lectures on Deformation Theory

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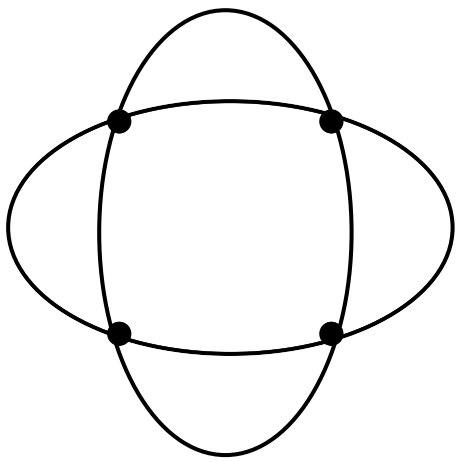
ghostview

Chapter 1

Introduction to Deformation Theory

Example 1.1 (Plückers idea, (1839)). Let $X = V(f) \subset \mathbb{P}^2$, with $f \in \mathbb{R}[x, y, z]$ be a curve in the plane . What can one say about the shape of X? J. Plücker used deformation theory in order to get an idea of the possible shapes of a quartic curve.
Take $f = Q_1 \cdot Q_2$ be the product of two general quadrics Q_1 and Q_2 intersecting in four points. Its zero set looks like:



The four intersection points we call p_1, p_2, p_3, p_4. These points are called, for obvious reasons, *double points* of X. Now consider the polynomial

$$F = Q_1 \cdot Q_2 + s \cdot P \in \mathbb{R}[s, x, y, z]$$

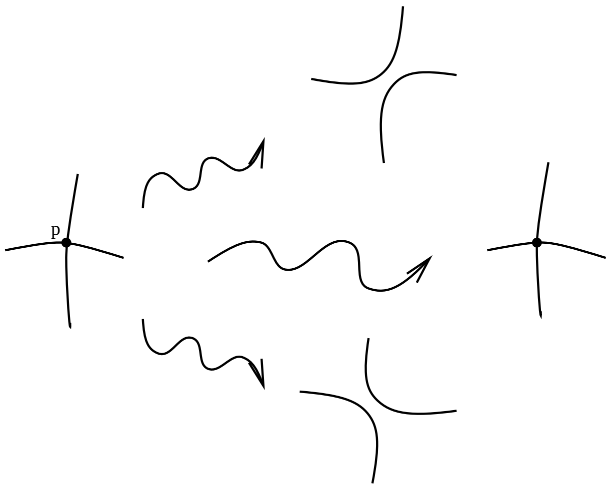
with P any homogeneous polynomial of degree 4 and put

$$X_S = V(F) \subset \mathbb{P}^2 \times S$$

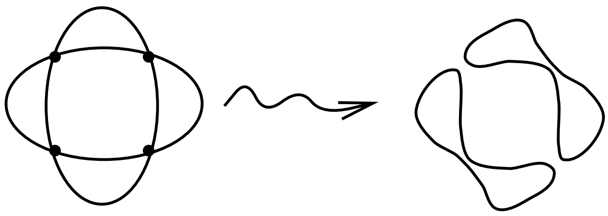
with $S = \mathbb{R}$. The map $(s, x, y, z) \mapsto s$ restricts to a map

$$\pi : X_S \longrightarrow S$$

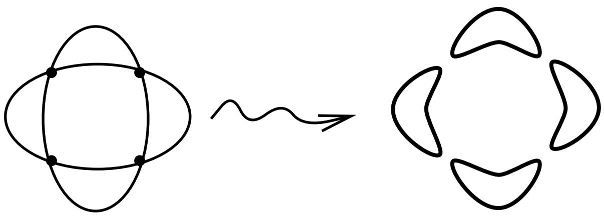
So X_S can be seen as a *family* of curves $X_s := \pi^{-1}(s)$. When $|s| << 1$, the curve X_s will be very close to our original curve $X = X_0$. We investigate what can happen locally at a double point p. There are three possibilities, indicated by the following pictures



It is a non-trivial fact that one can deform each of the double points *independently in any of the above three ways.* For example, one can obtain a quartic with this shape:



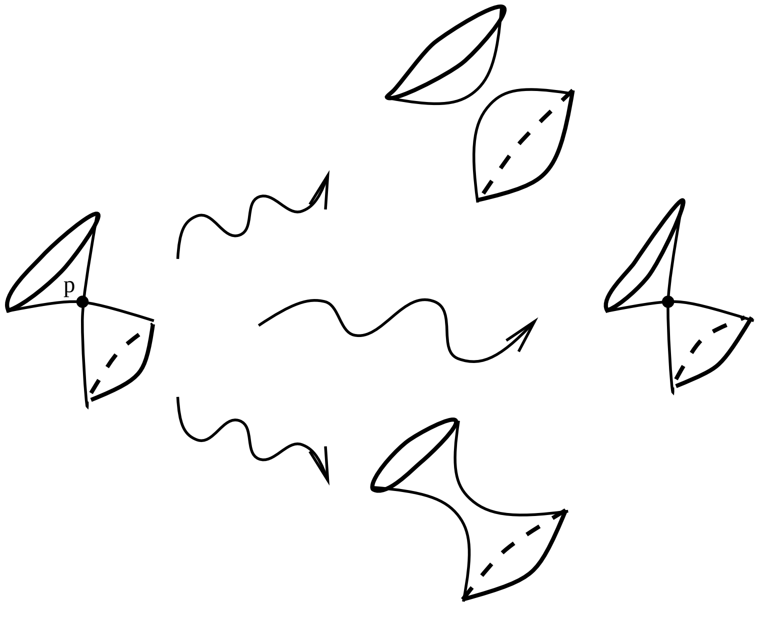
or the well-known quartic with four kidney shaped components.



Hence the statement is that by choosing appropriate perturbations P, one can create $3^4 = 81$ topologically distinct curves X_s near X.

Around 1880 Klein had the idea to do the same with surfaces in $\mathbb{P}^3$. For example, take the four nodal quartic $X = V(f)$, with $f = xyz + xyt + xzt + yzt$.

The singular points of X are ordinary double points. In an analogous way, we get by perturbation a family of surfaces $X_S \longrightarrow S$. Locally, near each of the double points, three things can happen.



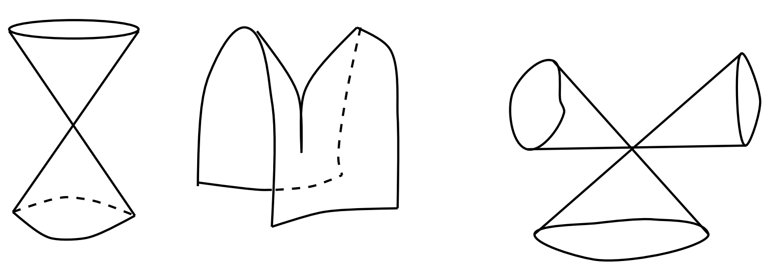
Klein showed that in this way one can generate all possible types of real cubics in three-space.

We want to stress here the fact that it is *not clear at all* that the local deformations around the singular points can be globalised to deformations of the whole surface. In fact, for more complicated examples this will not be the case. Later we will develop tools to handle such questions.

Example 1.2 (A-D-E-singularities). The classification of singularities of hypersurfaces up to right equivalence starts with the celebrated A-D-E singularities.

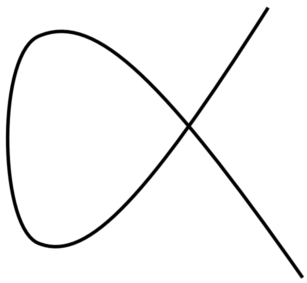
name	$f \in \mathbb{C}\{x,y\}$	
A_k	$y^2 - x^{k+1}$	$k \geq 1$
D_k	$x(y^2 - x^{k-2})$	$k \geq 4$
E_6	$y^3 - x^4$	
E_7	$y^3 - x^3 y$	
E_8	$y^3 - x^5$	

The germ A_0 is smooth, A_1 is usually called *ordinary double point*, A_2 the *cusp*, A_3 the *tacnode*, etc. The A-D-E surface singularities are obtained by adding a square in a new variable, $F = f + z^2$, similarly for threefolds, etc.

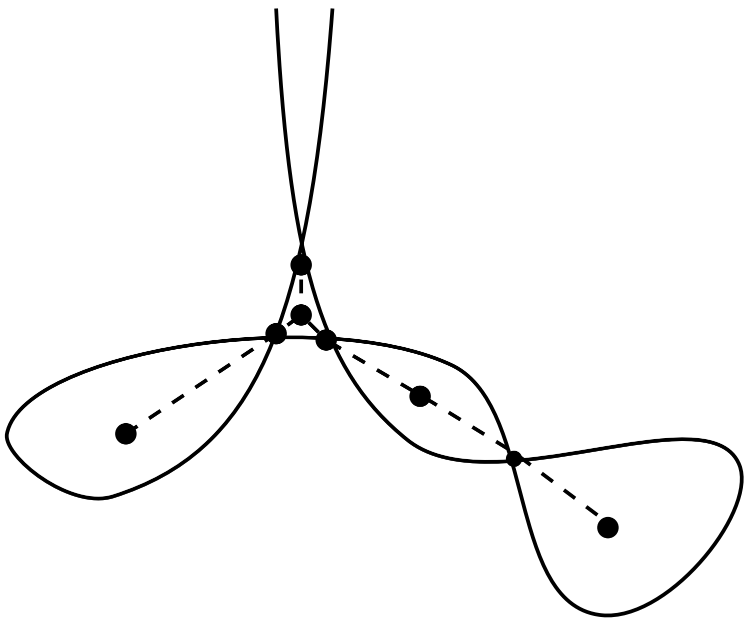


A_1 A_2 D_4

The name of these singularities come from the relation with the Dynkin diagrams with same name. Consider the a parametrisation $\phi : \mathbb{C} \longrightarrow \mathbb{C}^2 \ \ t \mapsto (t^3, t^2)$ which has the cusp $V(x^2 - y^3)$ as image. We can perturb the parametrisation to $\phi_S : \mathbb{C} \times S \longrightarrow \mathbb{C}^2 \times S \ \ (t, s) \mapsto (t^3 - ts, t^2)$ Now the image is $V(x^2 - y^3 + 2sy^2 - s^2y)$



We can do something similar with any of the other singularities in the list. For example E_8 $t \mapsto (t^5, t^3)$ is perturbed to $\phi_S(t, s) = (t^5 + sP_1, t^3 + sP_2)$ When we make an appropriate choice of P_1 and P_2 the image can look like:



The name of these singularities come from the relation with the Dynkin diagramswith same name. We will see later that a singularity with diagram D deforms into singularity with diagram D' if and only if D' is a subdiagram of D.

Example 1.3. Consider a varieties X and X_S defined by ideals

$$I = (f_1, f_2, \ldots, f_r) \subset k[x_1, x_2, \ldots, x_n]$$

and

$$I_S = (F_1, F_2, \ldots, F_r) \subset k[s, x_1, x_2 \ldots, x_n]$$

If $f_i(x) = F(0, x)$ we will say that X_S is a *deformation* of X. There is a canonical map $\pi : X_S \longrightarrow S$, with fibres X_s. The fibre over $s = 0$ is our original variety X. This state of affairs is usually depicted by the following diagram

$$\begin{array}{ccc} X & \hookrightarrow & X_S \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & S \end{array}$$

We also say that X_S is a *family* with fibres X_s.

We illustrate this concept with the following examples

(1) $X = V(f), f \in k[x_1, \dots, x_n]$, $X_S = V(F), F = f + s.g \in k[s, x_1, \dots, x_n]$ The above examples were of this type.

(2) $X = V(f_1, f_2)$, $f_1 = xy, f_2 = x^2 + y^2 - z^2$, $X_S = V(F_1, F_2)$, where $F_1 = f_1 + s$, $F_2 = f_2 + s$. The deformation $X_S \longrightarrow S$ decribes a family of curves in three space. More generally, if $\operatorname{codim} X =$ number of equations, we say that X is a *complete intersection.*

(3) $X = V(I), I = (f_1, f_2, f_3) = (yz, zx, xy) \subset k[x, y, z]$, $X_S = V(I_S), I_S = (F_1, F_2, F_3) = (yz - s, zx - s, xy - s) \subset k[s, x, y, z]$. The space X consists of the three coordinate axes. The fibres X_s consist of two points $\pm(\sqrt{s}, \sqrt{s}, \sqrt{s})$. So in this deformation the dimension of the fibre has changed. The reason is that the *relations* $xf_1 - yf_2$, $xf_1 - zf_3$, $yf_2 - zf_3$ do not extend to similar relations between $F_1, F_2 and F_3$. The deformation $X_S \longrightarrow S$ is not *flat.* The important concept of flatness will be explained in 7.

Example 1.4. Not everything that looks like a family is a family!!! Consider a map $\phi : \mathbb{C} \longrightarrow \mathbb{C}^3; t \mapsto (t^3, t^4, t^5) = (x, y, z)$ Let X be the image of this map. It is an exercise to show that $X = V(f_1, f_2, f_3) = V(xz - y^2, yz - x^3, z^2 - x^2y)$ Now consider the family of maps

$$\phi_S : \mathbb{C} \times S \longrightarrow \mathbb{C}^3 \times S; (t, s) \mapsto (t^3, t^4, t^5 + st^2) = (x, y, z)$$

Put $X_S := Im(\phi_S)$. Equations for X_S:

$$xz - y^2 = st^5; \quad yz - x^3 = st^6 = sx^2; \quad z^2 - x^2y = 2st^7 + s^2t^4 = xsxy + s2y$$

What to do with the term st^5? We cannot express it as an element of the ideal (x, y, z), but we can do the following:

$F_1 :\ x(xz - y^2) = st^8 = sy^2$

$G_1 :\ y(xz - y^2) = st^9 = sx^3$

$H_1 :\ z(xz - y^2) = st^5(t^5 + st^2) = sx^2y + s^2xy$

We need *five* equations to describe the image of ϕ_S to wit

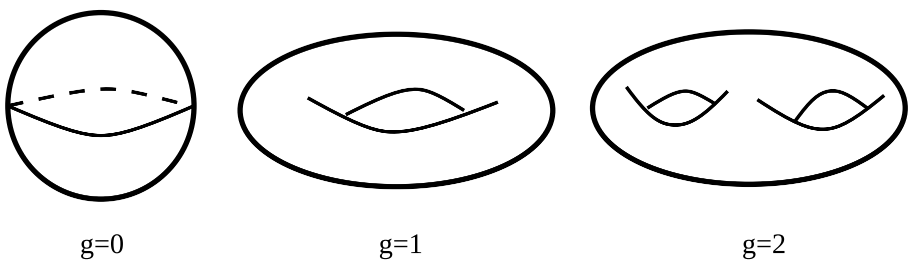
$$F_1, G_1, H_1, F_2, F_3$$

with $F_2 = yz - x^3 - sx^2$ and $F_3 = z^2 - x^2y = 2sxy - s^2y$. If we put $s = 0$ we see that the equations specialize to $xf_1, yf_1, zf_1, f_2, f_3$. So we see that $X_S \longrightarrow S$ is *not* a deformation of X!

Chapter 2

Riemann surfaces

The main discrete invariant for smooth compact curves (Riemann surfaces) is the *genus*. The genus is a topological invariant, which gives "the number of holes".



Curves of genus 0 are all isomorphic to $\mathbb{P}^1$: they are called rational curves. They can be embedded in $\mathbb{P}^2$ by an (affine) equation of type:

$$E_\lambda : y^2 = x(x-1)(x-\lambda).$$

The projection from the point $(0:0:1)$ on the x axis exhibits the elliptic curve as a $2:1$ covering of the x-axis branched over $0, 1, \infty$ and λ. On the other hand, every $2:1$ covering of $\mathbb{P}^1$ branched over four points gives an elliptic curve. Permuting the branch points gives that the isomorphism class of the elliptic curve is unchanged if one replaces λ by either

$$1-\lambda, 1/\lambda, 1/(1-\lambda), \lambda/(1-\lambda) \quad \text{and} \quad (\lambda-1)/\lambda.$$

The j invariant

$$j(\lambda = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

therefore classifies the elliptic curves up to isomorphism. There exist therefore a one parameter family of elliptic curves. Similar considerations we have for hyperelliptic curves. These are by definition (non-rational) curves which admit a $2:1$ covering of $\mathbb{P}^1$, i.e. have an affine equation of type $y^2 = f(x)$. Such a $2:1$ map, if it exists, is determined up to a automorphism of $\mathbb{P}^1$. If the genus is g, the number of branch points is $2g+2$ by the Hurwitz formula.

It follows that there is a $2g-1$ family of hyperelliptic curves of genus g. As every curve of genus 2 is hyperelliptic (the canonical system gives such a map), the curves of genus two form a three dimensional family.

We now consider arbitrary curves, and count the number of parameters. By Riemann-Roch they admit a finite map of degree at most $g+1$. For our purposes though, it will be better to consider $n-$ branched covers, for $n > 2g$. Any curve of genus g is an n-branced covering of $\mathbb{P}^1$. In fact, take any divisor D of degree n. From Riemann-Roch it follows that there exists a function $f \in L(D) = \{(f) + D \geq 0\}$ with pole divisor *exactly* D. Because the divisor $K - D$ has negative degree, it follows that the dimension of $L(D)$ is $n + g - 1$ This gives a map to $\mathbb{P}^1$. Divisor of degree n form an n dimensional family. The maps to $\mathbb{P}^1$ from a fixed Riemann surface therefore form a family of dimension $2n + g - 1$. As a curve of genus $g \geq 2$ can only have a *finite* number of automorphisms, it follows that there is only a zero dimensional family of morphisms from a given curve to $\mathbb{P}^1$ with given branch locus.

The number of branch points is by the Hurwitz formula equal to $2n + 2g - 2$. On the other hand, given a branch locus B, there exist finitely many Riemann-surfaces which are $n-$ branched covers of $\mathbb{P}^1$ and having branch locus B. The Riemann-surfaces therefore form a family of dimension $2n + 2g - 2 - (2n + g - 1) = 3g - 3$.

Another way to see this number for low genus is by considering the canonical embedding of degree $2g - 2$ in $\mathbb{P}^{g-1}$. For curves of genus $g \geq 3$ the canonical linear system defines an embedding precisely when the curve is non-hyperelliptic. We already saw that the the hyperelliptic curves form a family of dimension $2g - 1$. For example, plane curves of degree four in $\mathbb{P}^2$ have genus three, and are canonical. The quartic curves form a space of dimension 14, and the space of projective transformations of $\mathbb{P}^3$ is 8. Therefore, curves of genus three form a family of dimension 6. One shows that a canonical curve of genus 4 in $\mathbb{P}^3$ is the complete intersection of a quadric and a cubic surface in $\mathbb{P}^3$. Conversely, the adjunction formula gives that the intersection of a quadric and a cubic is a curve of genus 4. Counting parameters again, one finds that curves of genus four form a 12 dimensional family. Similarly one treats the case of canonical curves of genus 5 in $\mathbb{P}^4$.

Let us suppose that there exist a "universal" family of curves of genus g, $\mathscr{C} \to \mathscr{M}_g$. (This at least exists (for $g \geq 2$) for those curves which have no automorphism: but in fact for the following we just need the semi-universality of the family, as to be defined later.) How to compute the dimension of $\mathscr{M}_g$? Well, take a smooth point p of $\mathscr{M}_g$ (assuming that it exists), corresponding to a Riemann-surface X, and compute the dimension of the tangent space at this point. But the tangent space correspond to maps from the double point

$$\mathbb{T} := Spec(\mathbb{C}[\epsilon]) \to \mathscr{M}_g$$

sending the closed point to p. (We will always assume $\epsilon^2 = 0$.) We can restrict our universal family to the double point, and get a family

$$X_{\mathbb{T}} \to \mathbb{T}$$

with special fibre X. So the question becomes: How to classify these families? We take a (finite) open cover

$$X = \cup_{i=1}^n U_i$$

where each U_i is isomorphic to the unit disc. The idea (due to Kodaira and Spencer) is to take a covering:

$$X_{\mathbb{T}} = \cup_{i=1}^n (U_i \times \mathbb{T})$$

But to define $X_{\mathbb{T}}$ we need the transition functions. Let us spell this out: for each i we have local coordinates z_i which give an isomorphism between U_i and a unit disc Δ_i: The transition functions $f_{ij} := z_j z_i^{-1}$ are holomorphic on the domain of definition. Of course, whenever defined, we have

$$f_{ik} = f_{ij} f_{jk}$$

We peturb this suituation, i.e. we look at transition functions F_{ij} which now are depending on z_j and ϵ, and such that for $\epsilon = 0$ we get back our f_{ij}. We have the condition that on $U_i \cap U_j \cap U_k$:

$$F_{ik}(z_k, t) = F_{ij}(F_{jk}(z_k, \epsilon), \epsilon)$$

Writing $F_{ij} = f_{ij} + \epsilon g_{ij}$ we get by the chain rule the equation between tangent vectors:

$$g_{ij} \frac{\partial}{\partial z_i} = g_{ik} \frac{\partial}{\partial z_i} + \frac{\partial z_i}{\partial z_j} g_{jk} \frac{\partial}{\partial z_i}$$

But the last term is just the vector field $g_{jk}\frac{\partial}{\partial z_j}$ Therefore, if we define the vector field on $U_i \cap U_j$ by

$$\theta_{ij} := g_{ij}\frac{\partial}{\partial z_i}$$

we have that these satisfy the cocycle condition:

$$\theta_{ij} - \theta_{ik} + \theta_{jk} = 0$$

It is boring to check that this element in first Cech cohomology group $H^1(X, \Theta_X)$ is independent of the choices made. On the other hand, given a cocyle $g_{ij}\frac{\partial}{\partial z_i}$ one defines a deformation over $\mathbb{T}$ by giving its transition functions $F_{ij} = f_{ij} + \epsilon g_{ij}$This deformation turns out to be trivial exactly when we have a coboundary.

Theorem 2.1. *The deformations of a Riemann surface X over $\mathbb{T}$ are classified by $H^1(X, \Theta_X)$.*

Remark that this argument works also for general compact complex manifolds.

We go back to our compact Riemann surface. The locally free sheaf Θ is the dual of the canonical sheaf. We therefore have to compute $H^1(X, K^{-1})$. By Serre duality, this space is dual to $H^0(X, K^2) = L(2K)$. If the genus is zero, then the degree of $2K$ is negative, therefore $L(2K) = 0$. This says that $\mathbb{P}^1$ is rigid. This is as hoped for: the only curve of genus zero is $\mathbb{P}^1$. For genus one one has that K, and therefore $2K$ is trivial. Therefore, $L(2K)$ is one dimensional: elliptic curves form a family of dimension one. For higher genus we use Riemann-Roch:

$$l(2K) - l(K - 2K) = deg(2K) + 1 - g$$

Now $deg(K) = 2g - 2$, so $deg(2K) = 4g - 4$. Because $g \geq 2$, $deg(-K) < 0$, hence $l(-K) = 0$. So indeed we get $l(2K) = 3g - 3$.

Chapter 3

Deformation problems

Let us look at a problem from *linear algebra*, the classification of *matrices*. We consider the space

$$Mat := Mat(n \times n, \mathbb{C}) \approx \mathbb{C}^{n^2}$$

of $n \times n$ matrices over a field $\mathbb{C}$. We call two matrices *equivalent* if the matrices become the same after a change of base. There is an action of the group $GL(n, \mathbb{C})$, operating on Mat by change of bases:

$$GL(n, \mathbb{C}) \times Mat \longrightarrow Mat \ , \ (G, A) \mapsto GAG^{-1}$$

The equivalence classes of matrices are the orbits of this group action.

Normal Form Problem
Find a *representative* in each equivalence class (=group orbit). Of course, this problem is " solved " by the Jordan normal form.

Moduli Problem
Find a *space* whose points represent the equivalence classes.

Example 3.1. We look at the family of matrices $\begin{pmatrix} \alpha & s \\ 0 & \alpha \end{pmatrix}$ paramatrized by s: For $s = 0$ we have Jordan normal form $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, whereas for $s \neq 0$ we have Jordan normal form $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$. The family of matrices $\begin{pmatrix} \alpha & * \\ 0 & \alpha + s \end{pmatrix}$ has Jordan normal form $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha + s \end{pmatrix}$ if $s \neq 0$.

Suppose there would exist a "moduli–space" $\mathscr{M}$, with classifiying map

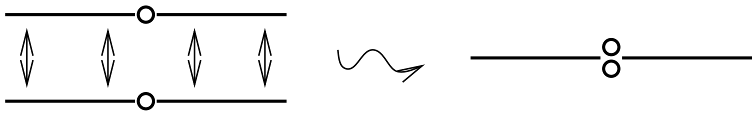
$$Mat(n \times n) \longrightarrow \mathscr{M}.$$

Consider the following two curves in $Mat(2 \times 2)$:

$$A_s = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha + s \end{pmatrix}.$$

$$B_s = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha + s \end{pmatrix}.$$

For $s \neq 0$ we see that $A_s \approx B_s$ The picture is at follows:



The left hand side shows the two curves in $Mat(2 \times 2)$. The arrows indicate that corresponding points become identified in $\mathscr{M}$, leading to the picture on the left hand side for $\mathscr{M}$. Therefore, if the moduli–space $\mathscr{M}$ would exist, it would be non–Hausdorff.

Problem 3.2 (Normal Form Problem for Families). Find good normal forms for *families of matrices.*

Definition 3.3.

- A *family of matrices* over S, where S is a complex space, is a holomorphic map
$$A : S \longrightarrow Mat$$
It is useful to use the notation A_S to denote such a family, and write $A_s := A(s)$, which we call *fibres* of the family.

- If $0 \in S$ is point, we call A_S a *deformation* of A_0. Alternatively, we say that A_0 is a *degeneration* of the *general fibre* A_s, $s \in S$.

Most of the time, we are only interested in the behaviour of families near $s = 0$.

Definition 3.4.

- Two deformations A_S and A'_S of A_0 are called *equivalent*, if there is a deformation G_S of the identity matrix $T_0 = I_n$, such that
$$A'_S = G_S A_S G_S^{-1}.$$
That is, for all $s \in S$
$$A'_s = G_s A_s G_s^{-1}.$$

- If $\phi : T \longrightarrow S$ is a map, and A_S is a family over S, representent by a map $S \longrightarrow Mat$, then the *induced family* $\phi^* A_T$ is just the composition $T \xrightarrow{\phi} S \longrightarrow Mat$. So, $\phi^* A_T$ is the family over T with fibre
$$\phi^* A_t = A_{\phi(t)}$$

Definition 3.5.

- A deformation $A_S \longrightarrow S$ of A_0 is called *versal* if *every* other deformation of A_0 is equivalent to one induced from A_S.

- A versal deformation is called *universal*, if this inducing map is uniquely determined.

- A versal deformation is called *miniversal* if it is versal of minimal dimension.

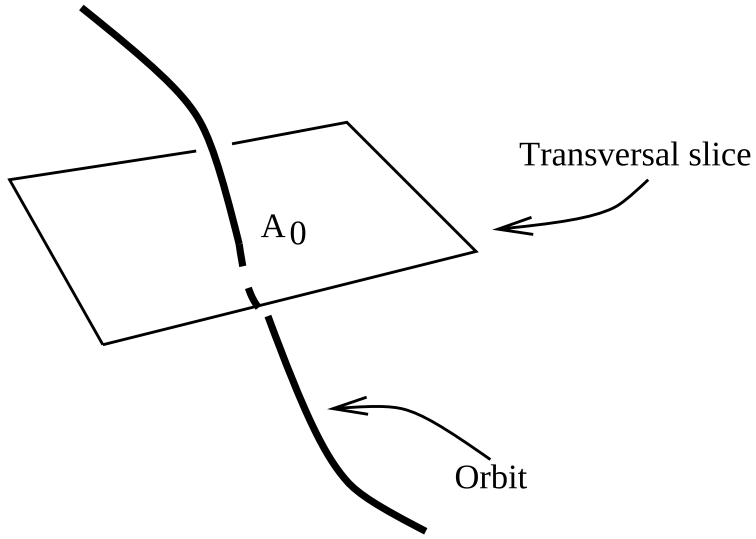
Example 3.6.

- Let $A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any matrix. The space of all 2×2 matrices is a versal deformation of A_0.

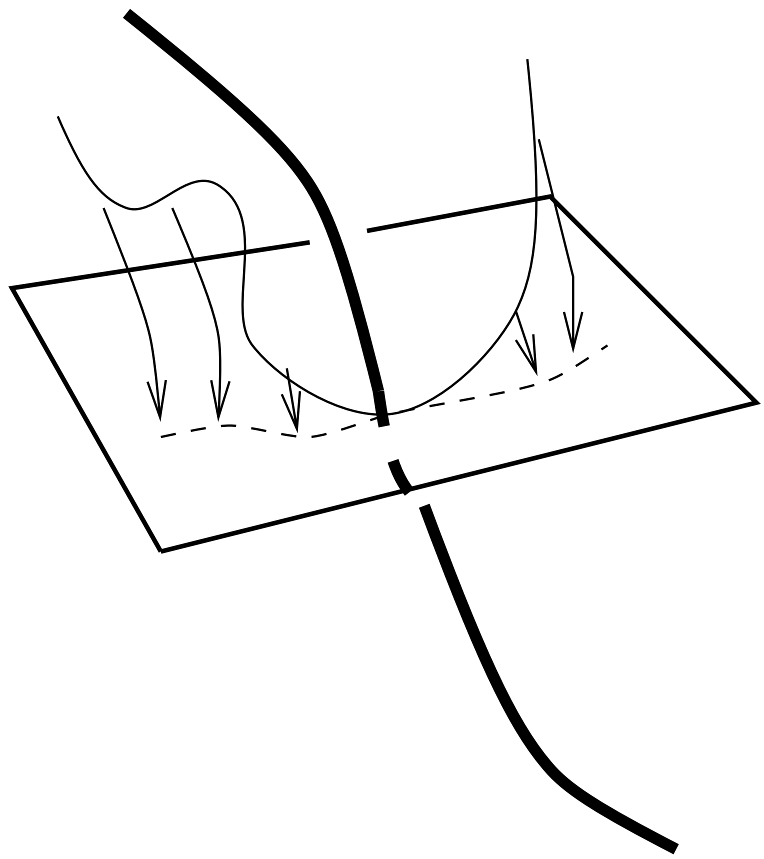
- $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix}$ is a versal deformation. It is even universal.

- $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}$ is miniversal.

There is a simple geometrical method to obtain a versal deformation of a matrix A_0: look at the $GL(n)$-orbit of the matrix A_0. Now take a transversal slice to the orbit.



Intuitively it is clear that any family can be transformmed into a given transversal slice, using the group action:



Chapter 4

Surfaces I

In the theory of compact complex smooth curves, we have a division in three cases:

- $g = 0$. The rational curve $\mathbb{P}^1$.
- $g = 1$. Elliptic curves
- $g \geq 2$ Curves of "general type".

Here g is the number of independent 1–forms on the curve. The Euler number (Euler characteristic) e is related to the genus: $e = 2 - 2g$.

But for surfaces one has different generalizations.

(1) Euler Characteristic of X . This is very computational, in fact it usually suffices to use the following "axioms":

   (a) $e(X \cup Y) = e(X) + e(Y)$
   (b) $e(X \times Y) = e(X) \cdot e(Y)$
   (c) $e(P) = 1; e(I) = -1$, where I is the open interval.

(2) The canonical class. One has the following "adjunction formulas":

   (a) Let $X \subset Y$ be a smooth hypersurface. Then

   $$K_X = (K_Y + X)X$$

   This can be considered as $l'Hopital's$ rule.

   (b) If X is a blow-up of Y, with exceptional divisor E, then:

   $$K_X = f^*(K_Y) + E$$

   (c) If $f : X \longrightarrow Y$ is a double cover of Y, branched over $B \subset Y$ then

   $$K_X = f^*(K_Y + B)$$

   Numerical invariants one can extract from the canonical class are

   - K^2
   - $\dim H^0(K) = p_g$ the "number of two–forms", also called the geometric genus.

We emphasize that e and K^2 are relatively easy to compute. In fact, these are nothing but the familiar chern numbers of the tangent bundle of the surface.

Remark 4.1. $e = c_2(\theta)$; $K = -c_1(\theta)$,. Therefore $e = c_2$ and $K^2 = c_1^2$.

The computation of p_g is more subtle.

Further invariants:

(1) Hodge numbers: $h^{p,q} = \dim H^q(X, \Omega^p)$, $h^{p,q} = h^{q,p}$. These make up the Hodge-diamond, which for surfaces looks like

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{11} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

The number of holomorpic one-forms $q = h^{10} = \dim H^0(\Omega^1)$ is called the irregularity of the surface.

(2) On $H^2(X, \mathbb{Z})$ we have the intersection form:

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow H^4(X, \mathbb{Z}) = \mathbb{Z}$$

which is a uni–modular quadratic form, whose rank is $2p_g + h^{1,1}$. The index of this quadratic form (the number of positive minus the number of negative eigenvalues) is equal to $\frac{c_1^2 - 2c_2}{3} =: \tau$, and therefore:

$c_1^2 = K^2$ is a topogical invariant!

In the simply connected case (irregularity is zero) we get that the rank is equal to $c_2 - 2$. It is a deep theorem of Friedman that rank and signature of the intersection form, and hence e and K^2, is a complete topological invariant!

One has the following "rough classification" of surfaces:

- K "negative": $H^0(nK) = 0$ for $n >> 0$
- K "zero": $H^0(nK)$ is bounded
- K "positive": $H^0(nK) \to \infty$

We now consider this first case $\boxed{K \text{ negative}}$

If K is negative, then there is a rational curve C on the surface with $C^2 = 0$. Then the surface has a $\mathbb{P}^1$–fibration. This is an example of deforming a curve in a surface, see ???

Riemann-Roch for Surfaces:

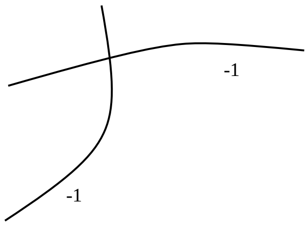
$$\chi(\mathscr{O}(D)) := h^0(D) - h^1(D) + h^2(D) = \frac{1}{2} D(D - K) + \chi(\mathscr{O})$$

were $\chi(\mathscr{O}) = 1 - q + p_g$. This can be computed from the Noether formula

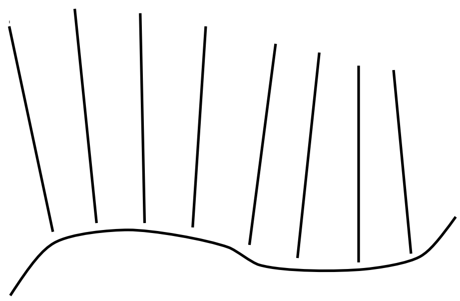
$$\chi(\mathscr{O}) = \frac{c_1^2 + c_2}{12} \ .$$

Serre-duality tells us: $h^2(D) = h^0(K - D)$.

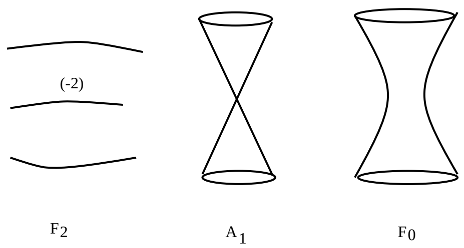
Suppose X is a surface with two intersecting (-1)–curves, as suggested in the following picture:



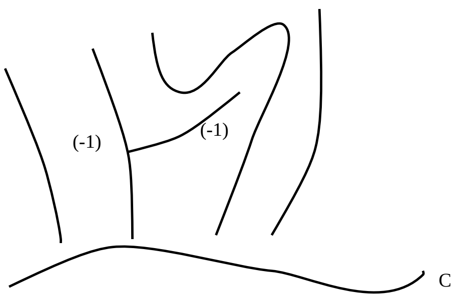
We have $KC + C^2 = -2$, $h^0(C) + h^2(C) \geq \chi(\mathscr{O}) + 1$, which implies that the curve C moves in a linear system $\Longrightarrow$ the surface is ruled. So if X is **not** ruled, (-1) curves do not intersect. You always can blow down (-1)-curves. After you have done that, you have what is called a **minimal ruled surface**. Such ruled surfaces without (-1) curves are $\mathbb{P}^1$ bundles over a curve C, and are of the type $\mathbb{P}(E)$, where E is a rank two vector bundle on C. E is determined up to tensoring with a line bundle L, that is $\mathbb{P}(E) \simeq \mathbb{P}(E \otimes L)$.



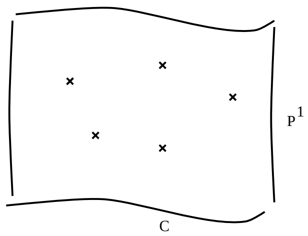
For example, for $C = \mathbb{P}^1$ and $E = \mathscr{O} \oplus \mathscr{O}(n)$ we get the Hirzebruch surface $\mathbb{F}_n$. These surfaces have the following peculiarity: the surface $\mathbb{F}_{n+2}$ deforms into $\mathbb{F}_n$, so $\mathbb{F}_n$ is diffeomorphic to $\mathbb{F}_m$ if $n = m$ mod 2. The deformation of $\mathbb{F}_2$ into $\mathbb{F}_0$ is the phenomenon of simultaneous resolution of the A_1 surface singularity, see ???:



In general, let us blow up a point on a fibre of a minimal ruled surface. We get two (-1)-curves:



We can blow down the other (-1) curve and get another minimal ruled surface. Hence from a ruled surface $X \longrightarrow \mathbb{P}^1$ we get $X' \longrightarrow \mathbb{P}^1$. The transition from X to X' is called an *elementary transformation*. One can perform a finite sequence of such elementary transformations that converts X into a product



The conclusion of all this is, that when the pluri-genus $P_n := H^0(nK) = 0$ for all $n \geq 0$, then X is birational to a product $C \times \mathbb{P}^1$. In fact one has:

- $P_{12} = 0 \Rightarrow P_n = 0$ for all n. If $q = 0$, then $P_2 = 0$ implies already $P_n = 0$. The surface X is rational.

Chapter 5

Surfaces II

Now we consider the case $\boxed{K \text{ zero}}$, meaning that $H^0(nK)$ remains bounded. In this case, there exist a covering $\widetilde{X} \longrightarrow X$ with $K = 0$, $h^0(nK) = 0, 1$.

A cubic $\mathbb{P}^2$ is an elliptic curve, $\mathbb{C}/\Lambda$ and as such admits two different generalisations to surfaces. We can consider quartic surfaces in $\mathbb{P}^3$. These are K3-surfaces and have $q = 0$. Or we can consider complex tori $\mathbb{C}^2/\Lambda$, which have $q = 2$. The complex torus has Euler number 0, but what is the Euler number of the Quartic in $\mathbb{P}^3$? We take a generic pencil of hyperplane sections X_t. Each X_t is a plane quartic curve. The generic X_t will be a smooth genus three curve, hence $e(X_t) = -4$. When a hyperplane of the pencil becomes tangent to the surface, we get a nodal quartic, which has $e(X_s) = -3$. Let n be the number of such nodal quartics. When we blow up the quartic in the points of intersection with the base locus of the pencil we get a surface X', fibred over $\mathbb{P}^1$, hence:

$$e(X') = e(X_t)(e(\mathbb{P}^1) - n) + n.e(X_s) = -4(2 - n) - 3n = n - 8$$

What is n? It is the number of intersection points of the surface with two generic polars, so $4.3.3 = 36$. We conclude that $e(X) = 24$.
One also can compute this from Noethers formula

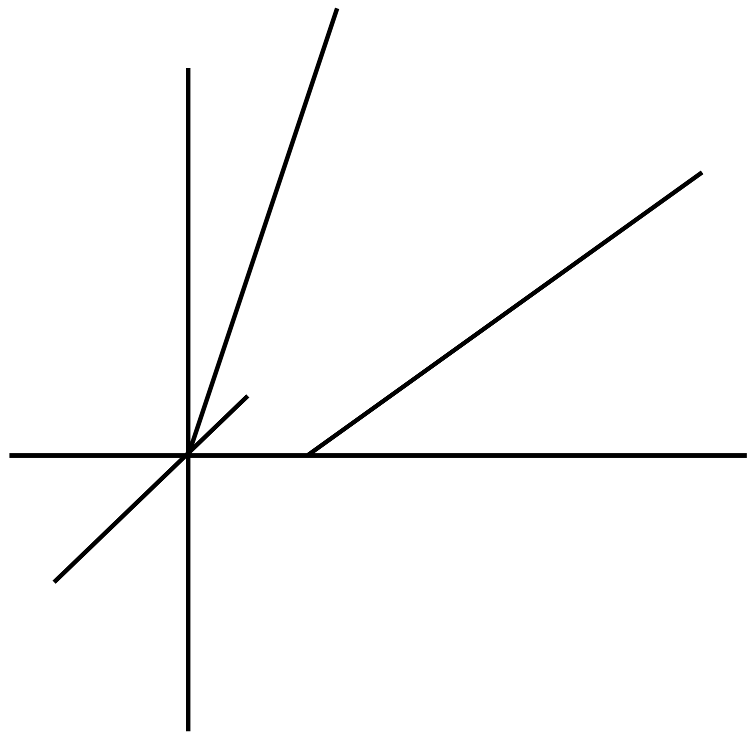
$$\frac{c_1^2 + c_2}{12} = \frac{c_2}{12} = 1 - q + p_g$$

As a smooth hypersurface in $\mathbb{P}^3$ is simply connected, one has $q = 0$. Moreover, $p_g = 1$ as $K_X = 0$, so we get $c_2 = 24$. There is an interesting relation between tori and $K3$-surfaces. When we divide the torus $\mathbb{C}^2/\Lambda$ by the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$, we get a surface X with 16 A_1-singularities. Resolving this gives us a $K3$ surface with 16 disjoint (-2) curves. Such surfaces are called Kummer surfaces, and all arise in this way.

Let us turn to the case $\boxed{K \text{ positive}}$. The first case is when $H^0(nK) \sim n$. In this case X admits a fibration with elliptic curves. In general, a surface X with a map $X \longrightarrow \mathbb{P}^1$ with generic fibre an elliptic curve E is called an elliptic surface. A deformation of such an elliptic surface is not necessarily elliptic. An elliptic surface has $K = rE$. The number r can be negative e.g. $\mathbb{P}^2$ blown up in 9 points. When we wiggle the points the elliptic fibration disappears: the surface is no longer elliptic. Honestly elliptic surfaces have $K = rE$ with $r > 0$. This class is stable under deformation.

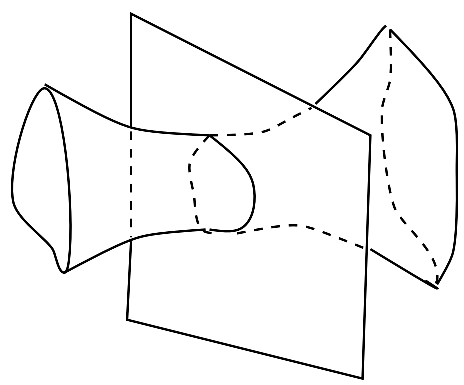
Rest of the surfaces: general type. Let us consider the pairs (c_1^2, c_2) for minimal surfaces (no (-1) curves). For these the following inequalities hold:

$$2\chi - 6 \leq c_1^2 \leq 3c_2 = 9\chi$$



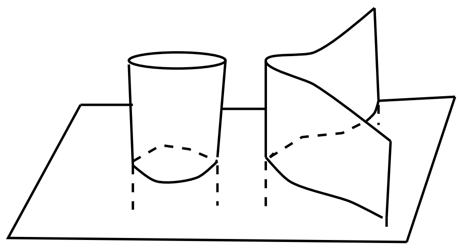
Ruled surfaces: $c_1^2 = 8(1-g)$, $\chi = (1-g)$
Godeaux surfaces: $c_1^2 = 1$, $\chi = 1$.

How to constructs interesting surfaces? Look at surfaces of degree n in $\mathbb{P}^3$. The canonical class is $(n-4)H$. For $n < 4$ the surface is rational, for $n = 4$ we get $K3$ and for $n \geq 5$ we get surfaces of general type. Similarly, one can consider complete interesections in products of projective spaces, compute invariants, etc. Other examples arise by *imposing singularities*. We consider hypersurfaces with certain types of singularities and relate c_1^2 and c_2 of the minimal resolution with the general smooth surface. So let X_t degenerate into X_0. Assume for simplicity that X_0 has a single singular point p. Consider a resolution $\widetilde{X} \longrightarrow X_0$ of the special fibre. So we have $\pi^{-1}(X_0 - \{p\}) \approx \widetilde{X} - \pi^{-1}(p)$. If we let p be the point $(0:0:0:1)$ then the affine equation of X takes the form $F = F_m + F_{m+1} + \ldots$ where the F_i are homogeneous forms of degree i in x, y, z. m, the degree of the lowest order term is called the multiplicity of the singular point. If $m = 2$, and F_2 describes a non-degenerate quadric in three variables, then p is called an ordianry double point. When we blow up $\mathbb{P}^3$ at the point p, the exceptional $\mathbb{P}^2$ intersects the strict transform of the surface in the conic $F_2 = 0$.

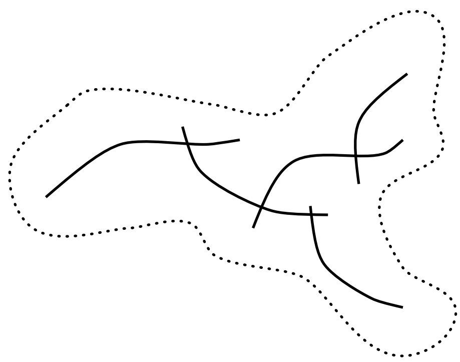


This curve has self-intersection -2 on the strict transform $\widetilde{X}$.

Similarly, when $m = 3$, the strict transform of the surface intersects the exceptional $\mathbb{P}^2$ in a cubic curve $F_3 = 0$. We speak of an ordinary triple-point if this cubic is smooth ($\widetilde{E}_6$- singularity, see ????)



In general, one can resolve the singularity by repeating this process of blowing up. In the end one arrives at a smooth surface, containing some configurations of curves that are contracted when mapped back into $\mathbb{P}^3$.



Knowing these resolution graphs, it is straightforward to relate $e(X_0)$ and $e(\widetilde{X})$. To see the relation between $e(X_0)$ and $e(X)$ we proceed as follows: take general coordinates so that $T = 0$ intersects the surface transversely along a smooth curve. The function $F(X, Y, Z, T)/T^n$ defines a fibration $\mathbb{P}^3 \setminus \{T = 0\} =: \mathbb{C}^3 \longrightarrow \mathbb{C}$, whose fibre over $c \in \mathbb{C}$ is the level surface $F(X, Y, Z, T) - cT^n$, which is nothing but the affine level set $f(X, Y, Z) = c$, where $f(X, Y, Z) = F(X, Y, Z, 1)$. For general c this will be smooth, but at the points where

$$\partial f/\partial X = \partial f/\partial Y = \partial f/\partial Z \ ,$$

the fibres will aquire singularities. If we count each such point with multiplicity

$$\mu = \dim \mathbb{C}[[x, y, z]]/(\partial_x f, \partial_y f, \partial_z f)$$

we have in total $(n-1)^3$ such points. We can assume that all singulartities not on the zero fibre are ordinary double points. From

$$1 = e(\mathbb{C}^3) = e(X)(2 - (n-1)^3) + (n-1)^3(e(X) - 1)$$

it follows that a singularity with Milnor number μ decreases the Euler number always by μ, so that

$$e(X_0) = e(X) - \mu.$$

The canonical divisor of X_0 is $-4H + nH|X_0 = (n-4)H|X_0$ by adjunction. The canonical divisor of $\widetilde{X}$ is given as

$$K_{\widetilde{X}} = K_{X_0} + \sum \alpha_i E_i$$

where i labels the exceptional curves in the resolution. The coefficients α_i can be computed using the adjunction formula for $E_i \subset \widetilde{X}$:

$$K_{\widetilde{X}} E_i + E_i^2 = 2g(E_i) - 2$$

This produces a system of linear equations for the α_i, which has a unique solution, because the matrix $(E_i \cdot E_j)$ is negative definite.

So we can compute $K_{\widetilde{X}}$ and hence the number $K_{\widetilde{X}}^2$.

Chapter 6

A–D–E singularities

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of an analytic function. Such germs form a ring, which we denote by $\mathscr{O}_n$. This ring $\mathscr{O}_n$ is in fact isomorphic to the power series ring $\mathbb{C}\{z_1, \ldots, z_n\}$. We want to classify singularities, up to a sensible equivalence relation. There are several possibilities:

Definition 6.1.

- f and g are called right equivalent, if there is an analytic automorphism $h \in \mathrm{Aut}(\mathbb{C}^n, 0)$ such that $f = g \circ h$, that is the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}, 0) \\ \big\downarrow{\scriptstyle h} & & \| \\ (\mathbb{C}^n, 0) & \xrightarrow{g} & (\mathbb{C}, 0) \end{array}$$

- f and g are called left-right equivalent, if there is an analytic automorphism $h \in \mathrm{Aut}(\mathbb{C}^n, 0)$ and an automorphism $\varphi \in \mathrm{Aut}(\mathbb{C}, 0)$ such that $\varphi \circ f = g \circ h$, that is the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}, 0) \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle \varphi} \\ (\mathbb{C}^n, 0) & \xrightarrow{g} & (\mathbb{C}, 0) \end{array}$$

- f and g are called contact equivalent, if there is an analytic automorphism $h \in \mathrm{Aut}(\mathbb{C}^n, 0)$ and a function $u\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^*, 0)$ (so u is a unit in $\mathscr{O}_n$ such that $u \cdot f = g \circ h$. This is equivalent to the condition that h maps the germ $(f^{-1}(0), 0)$ isomorphically onto $(g^{-1}(0), 0)$

In any case, we have a group G acting on $\mathscr{O}_n$, and we want to classify the orbits, similarly to the case of matrices in Chapter 3. We only have one problem:

$\mathscr{O}_n$ is an infinite dimensional vector-space

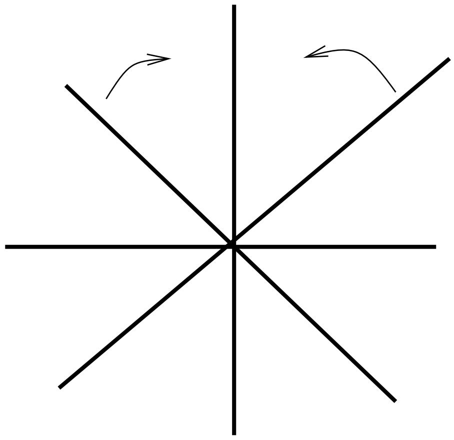
In the case of isolated singularities, one can circumvent this problem by using the finite determinacy theorem.

Theorem 6.2 (Finite determinacy Theorem). *Let $f \in \mathscr{O}_n$ have an isolated singularity. Then there exists a k , such that for all $g \in \mathfrak{m}^k$ the function $f + g$ is right-equivalent to f.*

In fact in the finite determinacy theorem, it suffices to take $k \geq \mu(f) + 1$, $\mu(f)$ is the Milnor number. This shows that for any particular f with isolated singularity, we may look at the induced group action on $\mathscr{O}/\mathfrak{m}^k$, which is a finite dimensional vector space.

Definition 6.3. $(X,0)$ is called *simple*, if such transversal slices intersects only finitely many orbits. Equivalently, $(X,0)$ deforms into finitely many isomorphism classes of singularities.

Example 6.4. Consider the singularity given by $xy(y^2-x^2)+\lambda x^4=0$



The cross-ratio of the lines changes. This leads to a continuous family of non-isomorphic singularities. (Moduli).

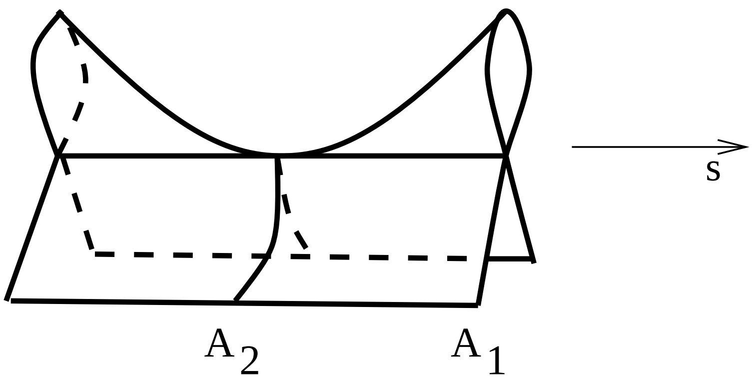
Theorem 6.5 (Arnol'd). *A Hypersurface singularity is simple if and only if it is of type A–D–E.*

Definition 6.6. $(X,0)$ is called *adjacent* to $(Y,0)$ if there exists a one–parameter family $X_S \to S$, $0 \in S$, such that

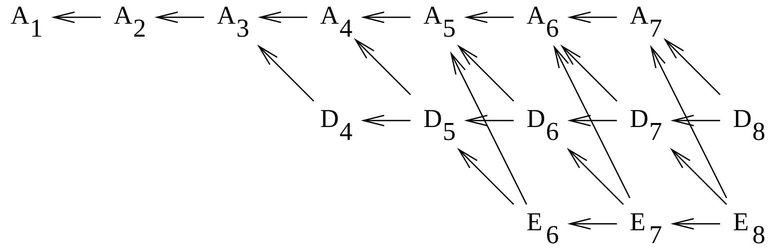
$$\begin{array}{rcll} (X_0,0) & \simeq & (X,0) & \\ (X_s,0) & \simeq & (Y,0) & \text{for } s \neq 0 \text{ small.} \end{array}$$

Notation: $(X,0) \longrightarrow (Y,0)$.

For example, we have $A_1 \longleftarrow A_2$, as the formula $y^2-x^2(x-s)$ shows. This is illustrated by the following picture:



The following diagram gives the all the adjacencies for the simple singularities.

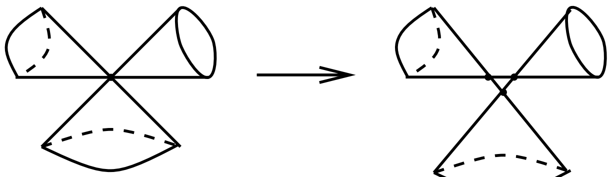


More generally, one can ask how many singularities, and of which type, might appear on a general fibre of a deformation of a (simple) singularity. In general, this is a very difficult question, but for simple singularities there is a beautiful answer.

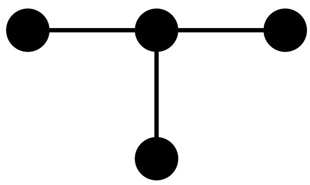
Example 6.7. Consider the deformation given by the following equation:

$$(y - s)(x^2 - y^2) - z^2 = 0$$

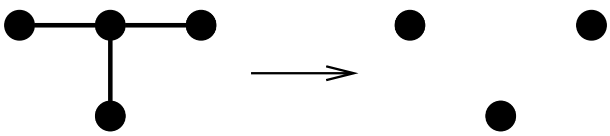
where s is the deformation parameter. It has on the zero fibre one D_4 singularity, on the general fibre there are s A_1 singularities. The picture is at follows.



The answer to the question above is in terms of the Dynkin diagrams. We proceed with the above example. The Dynkin diagram is:



By throwing away some vertices, and all edges which are adjacent to these vertices, one gets a (in general non connected) graph wiith say p components. This graph we may interprete as p Dynkin diagrams. In our example of the D_4 we delete the middle vertex and the corresponding edges:



In this way we get the Dynkin diagram of three A_1 singularities.

Theorem 6.8. *Consider a A–D–E singularity $(X, 0)$. Let $X_S \longrightarrow S$ be a 1-parameter deformation of $(X, 0)$, with (simple) singularities $X_1 \ldots X_p$ on the general fibre. Let $\Gamma_i, i = 0, \ldots, p$ be the Dynkin diagram of X_i. Then it is possible by deleting some vertices of Γ_0 and the adjacent edges to these vertices to get a graph Γ with p components $C_1, \ldots, D_p$, such that C_i is the Dynkin diagram of X_i for $i = 1, \ldots, p$.*

Conversely, if one has such an operation on the graph there exists a 1-parameter deformation of $(X, 0)$ with corresponding singularities in the general fibre.

A (conceptual) proof of this theorem will be given in **??**

Chapter 7

Flatness

In this part we consider *singularities* (that is, germs of analytic spaces) and their deformations. Such a singularity is is given as the zero-set of a set of analytic function:

$$f_1(x) = \ldots = f_k(x) = 0$$

We consider *deformations*

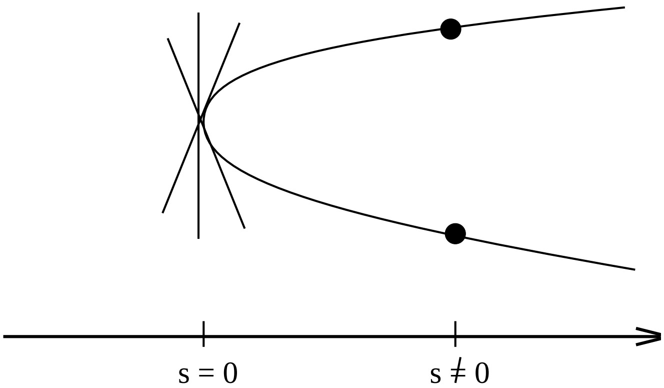
$$F_1(x,s) = \ldots = F_k(x,s) = 0$$

where $F_i(x,0) = f_i(x)$ for $i = 1, \ldots k$. We start with recalling the example in 1.3 of a family which is not flat.

Example 7.1. Let $X_S \longrightarrow S$ be a a family which is defined by

$$\begin{array}{rcccl} F_1 & = & xy - s & = & 0 \\ F_2 & = & xz - s & = & 0 \\ F_3 & = & yz - s & = & 0 \end{array}$$

Although this is a variety defined in 4-space, we would like to imagine the situation by the following picture:



The problem here is that s is zero-divisor of $\mathbb{C}\{x,y,z,s\}/(F_1,F_2,F_3)$. The total space X_S has four components, of which three are in the fibre $s = 0$. This fibre consists of the three coordinate axes. We can therefore decompose X_S:

$$X_S = X_0 \cup X_1$$

where X_0 are the three coordinate axes, and X_1 is the parabola. Take a function f, vanishing on X_1, but not on X_0 (the existence of such a function goes under the name "prime avoidance"). Then obviously:

$$s \cdot f = 0 \quad \text{on } X_S \text{ but } s, f \neq 0 \in \mathcal{O}_{X_S}$$

expressing the fact that s is a zero-divisor. In fact the so-called active lemma of analytic geometry can be restated as:

Lemma 7.2. *s is zero-divisor $\iff$ there exists a component of X_S in the zero-fibre.*

The above example hopefully makes clear to the reader that for a "nice" one parameter family (with parameter s), one should impose the condition that s is a nonzero-divisor. This is called flatness:

Definition 7.3. A $\mathbb{C}\{s\}$-module M is called *flat* iff s is a nonzero-divisor of M.

For example a *finitely generated* $\mathbb{C}\{s\}$-module M is flat if and only if M is free. This is a direct consequence of the classification Theorem of finitely generated modules over a principal ideal domain.

It turns out that is not so easy at all to construct (non-trivial) one parameter *flat* deformations of singularities. It is natural to construct deformations by "power-series expansion. That is construct deformations over $\mathrm{Spec}(\mathbb{C}[s]/s^2)$, then try to lift to $\mathrm{Spec}(\mathbb{C}[s]/s^3)$, etcetera. But then one has the problem of defining, when a module is a flat $\mathbb{C}[s]/s^2$-module, as s is a zero-divisor of $\mathbb{C}[s]/s^2$. So we need to give a different definition of flatness of $\mathbb{C}\{s\}$-module, which gives a good generalization to rings with nilpotents. There are two reformulations.

Lemma 7.4. *Let S be the germ of a smooth 1-dimensional space and consider a deformation of X as above. Then the deformation is flat (that is, s is a nonzero-divisor) if and only if for every relation between the f_i:*

$$f_1 r_1 + \ldots + f_k r_k = 0$$

we can find a lift $R_i(x, s)$ *with* $R_i(x, 0) = r_i(x)$ *with*

$$F_1 R_1 + \ldots + F_k R_k = 0$$

Proof. The proof is elementary, but let us spell it out. Suppose that s is a nonzero-divisor, and take a relation $\sum f_i r_i = 0$. Take *any lift* $R'_i(x, s)$ of the r_i, and look at:

$$F_1 R'_1 + \ldots + F_k R'_k$$

This might not be zero, but we now it is if we plug in $s = 0$. This show that this expression is divisible by s:

$$F_1 R'_1 + \ldots + F_k R'_k = s\Phi$$

This expression says that $s\Phi = 0 \in \mathcal{O}_{X_S}$. As s is a nonzero-divisor it follows that $\Phi = 0 \in \mathcal{O}_{X_S}$, that is, $\Phi = \sum \alpha_i F_i$ for some α_i. Now put $R_i = R'_i - s\alpha_i$. It follows that

$$F_1 R_1 + \ldots + F_k R_k = 0$$

so we found a lift of the r_i. On the other hand, suppose that we can lift any relation. We need to show that s is a nonzero-divisor. So suppose that $s\Phi = 0 \in \mathcal{O}_{X_S}$, that is

$$s\Phi = F_1 R_1 + \ldots + R_k F_k R_k$$

Putting $s = 0$ we get a relation $\sum f_i r_i = 0$, which by assumption can be lifted to a relation $\sum F_i R'_i = 0$. Then

$$s\Phi = \sum F_i (R_i - R'_i)$$

As both R_i and R_i' are lifts of the r_i, it follows that $R_i - R_i'$ is divisible by s. Hence the power-series $\frac{R_i - R_i'}{s}$ exists. Because in the power-series ring s is a nonzero-divisor it follows that:

$$\Phi = \sum F_i \frac{R_i - R_i'}{s}$$

expressing the fact that $\Phi = \mathscr{O}_{X_S}$, which is what we had to show. □

In commutative algebra there is a different definition of flatness. To explain this, take R to be a commutative ring with 1, and let M be an R-module. Then M is called flat if for *all* exact sequences of R-modules:

$$0 \longrightarrow N' \longrightarrow N$$

the sequence

$$0 \longrightarrow N' \otimes M \longrightarrow N \otimes M$$

is also exact. To put it in another way, the functor $- \otimes M$ is a (left) exact functor.

Example 7.5. Take $R = \mathbb{C}\{s\}$, and M an R-module. Then

$$0 \longrightarrow \mathbb{C}\{s\} \longrightarrow \mathbb{C}\{s\}$$

is exact. If M is flat, it follows that

$$0 \longrightarrow M \xrightarrow{\cdot s} M$$

is exact, meaning that s is a nonzero-divisor of M.

For the case we are interested in, both notions of flatness coincide.

Chapter 8

The Language of Fibred Categories

We have been discussing by way of examples various types of *families* X_S over S and associated *deformation problems.*

(1) families of affine varieties over S.

(2) families of maps over S.

(3) families of Riemann surfaces over S.

(4) families of matrices over S.

(5) families of singularities over S.

(6) families of singularities, flat over S.

(7) families of schemes, flat over S.

(8) families of analytic spaces, flat over S..

(9) families of line bundles over S.

(10) families of curves in a given X over S..

We used notations like $X_S \longrightarrow S$ in each of these situations.

common feature The notion of *induced family*: Given X_S over S and $\phi : T \longrightarrow S$ is a map, then there exists something called $\phi^*(X_S)$, a family over T. In each of these cases one can formulate the notions of a versal deformation and the problem as to its existence can be posed.
There is a a precise but unspecific language that covers all these cases in a single formalism:

fibred categories

It sounds difficult, but it is not; as with all category stuff, it is basically "empty" [1]. We will have to deal with *two* categories. A category $\underline{\mathbf{F}}$, whose objects make up the *families* of objects we want to consider, and a category $\mathbf{C}$, whose objects correspond to the *parameter spaces* we have our families over. There is a *projection functor*

$$p : \underline{\mathbf{F}} \longrightarrow \underline{\mathbf{C}}$$

that assigns to each family $X_S \in Ob(\underline{\mathbf{F}})$ the parameter space $S \in Ob(\underline{\mathbf{C}})$ it is over.
Notation: Let $p : \underline{\mathbf{F}} \longrightarrow \underline{\mathbf{C}}$ a functor between categories.

[1] Siegel once refered to modern algebraic geometry in general as *the theory of the empty set*

- For $S \in Ob(\underline{\mathbf{C}})$ we put: $F(S) := \{X \in Ob(\underline{\mathbf{F}}) \mid p(X) = S\}$. This is the set of objects *over* S.
- For $\phi \in Mor(\underline{\mathbf{C}})$ we put: $F(\phi) := \{\psi \in Mor(\underline{\mathbf{F}}) \mid p_*\phi = \phi\}$. This is the set of morphisms *over* ϕ.

Definition 8.1. $p : \underline{\mathbf{F}} \longrightarrow \underline{\mathbf{C}}$ is called a fibred category if

(1) **Existence of pull-backs**: For all $\phi : T \longrightarrow S \in Ob(\underline{\mathbf{C}})$ and all $X_S \in F(S)$ there exists $\phi : X_T \longrightarrow X_S \in F(\phi) \subset Mor(\underline{\mathbf{F}})$

(2) **Strong uniqueness of pull-backs**: For all diagrams

$$\begin{tikzcd} T' \arrow[d] \arrow[dr] & \\ T \arrow[r] & S \end{tikzcd}$$

and $X_S \in F(S)$, there is a unique arrow $X'_T \longrightarrow X_T$ making a commutative diagram

$$\begin{tikzcd} S_{T'} \arrow[d] \arrow[dr] & \\ X_T \arrow[r] & X_S \end{tikzcd}$$

Corollary 8.2. *If we define the fibre category* $\underline{\mathbf{F}}(S)$ *as the category with objects* $F(S)$ *(the objects of* $\underline{\mathbf{F}}$ *over* S*, and morphisms* $F(Id_S)$ *(the morphisms in* $\underline{\mathbf{F}}$ *over the identity map,* $Id_S : S \longrightarrow S$*), then all morphisms in* $\underline{\mathbf{F}}(S)$ *are isomorphisms. Such a category is called a* groupoid.[2]

The language of fibred categories sets up the appropriate categorial way to discuss *families*. What about the categorial formulation of *deformation*? Let $X_0 \in Ob(\underline{\mathbf{F}})$, and put $0 = p(X_0) \in Ob(\underline{\mathbf{C}})$.

Definition 8.3. The *deformation category* of X_0 is the category $\underline{\mathbf{F}}_{X_0}$, which has

- $Ob(\underline{\mathbf{F}}_{X_0})$: morphisms in $\underline{\mathbf{F}}$ $X_0 \longrightarrow X_S$.
- $Mor(\underline{\mathbf{F}}_{X_0})$: diagrams

$$\begin{tikzcd} & X_S \arrow[d] \\ X_0 \arrow[ur] \arrow[r] & X_T \end{tikzcd}$$

The obvious functor $\underline{\mathbf{F}}_{X_0} \longrightarrow \underline{\mathbf{C}}$ represents $\underline{\mathbf{F}}_{X_0}$ as a fibred category. It has a special property, namely that the fibre category

$$\underline{\mathbf{F}}_{X_0}(0)$$

is a groupoid with one object, hence it is a *group*, to know, the group $Aut(X_0)$ of automorphisms of the object X_0 in $\underline{\mathbf{F}}$.

From the deformation category of an object X_0 one obtains a *deformation functor*

$$DF : \underline{\mathbf{C}} \longrightarrow (Sets)$$

[2] Any group can be made into a category with one object, and morphisms corresponding to the elements of the group, with composition in the category corresponding to multiplication in the group. A groupoid is a natural generalisation to categories with more objects. Every path connected topological space X has a fundamental groupoid: objects: points of X, morphisms from a to b: homotopy classes of paths from a to b. All groupoids are equivalent to such fundamental groupoids.

which associates to $S \in Ob(\underline{\mathbf{C}})$ the set $DF(S)$ of isomorphism classes of objects in $\underline{\mathbf{F}}_{X_0}(S)$. It is contravariant functor, because if $T \xrightarrow{\phi} S$ is a morphism in $\underline{\mathbf{C}}$, then we have a map $F(S) \longrightarrow F(T)$, by pulling back $X_S \in F(S)$ to $\phi^*(X_S) \in F(T)$.

In **??** we will discuss such functors more extensively.

In the literature one find often the dual notion of *cofibred category*, It is obtained by reversing arrows and is confusing, but more appropriate when one works with rings, rather than spaces.

Example 8.4. Cofibred category of Rings.

For deformation problems, there are five popular base categories of spaces $\underline{\mathbf{C}}$. It is easier to describe the opposite categories of rings. We fix a field k, which is $\mathbb{C}$.

(1) $\underline{\mathbf{C}}^{opp} = (Art)$, the category of artinian k-algebras.

(2) $\underline{\mathbf{C}}^{opp} = (\widehat{Art})$, the category of complete local k-algebras.

(3) $\underline{\mathbf{C}}^{opp} = (An)$, the category of analytic local rings.

(4) $\underline{\mathbf{C}}^{opp} = (Hens)$, the category of local henselian k-algebras.

(5) $\underline{\mathbf{C}}^{opp} = (loc)$, the category of local k-algebras.

From now on, we suppose that $\underline{\mathbf{C}}$ is one of these categories.

Definition 8.5. Let $p : \underline{\mathbf{F}} \longrightarrow \underline{\mathbf{C}}$ be a fibred categrory.

- An object $X_S \in F(S)$ is called *versal* if the following holds: for all $\phi : T \longrightarrow S$ and $\psi : T \hookrightarrow T' \in Mor(\underline{\mathbf{C}})$ and all $X_T \longrightarrow X_{T'} \in F(\psi)$, $X_T \longrightarrow X_S \in F(\psi)$, the following diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X_{T'} \arrow[dl, "\exists"] \\ X_S & \end{tikzcd}$$

  can be completed.

- X_S is called *formally versal* if the above condition holds for all $T \in T'$ in $(Art)^{opp}$ (which is a sub-category of each of the five popular base categories.)

- X_S is called *(formally) semi-universal* if

If $T = \{0\}$ the versality just says that all $X_{T'}$ over T' are induced by some map $T' \longrightarrow S$. So, from a versal family all other families can be induced. This is sometimes taken as a definition of versality, but of course in praxis one needs this stronger notion.[3]

[3]It would be interesting to know a geometrically meaningful example where the two notions differ.

Chapter 9

Schlessinger's Theorem

Let us consider a fibred category $p : \underline{\mathbf{F}} \longrightarrow \underline{\mathbf{C}}$ over the category $\underline{\mathbf{C}} = (Art)^{opp}$ of artin spaces. Given an $X_0 \in Ob(\underline{\mathbf{F}})$, $0 := p(X_0)$, we defined a deformation category $\underline{\mathbf{F}}_{X_0}$ and the associated deformation functor

$$F : (Art) \longrightarrow (Sets)$$

which associates a ring R from (Art) to the set of isomorphism classes of deformations of X_0 over $S = Spec(R)$.

One has $F(k) = [X_0]$, $F(k[\epsilon]) =$ deformations of X_0 over $\mathbb{T} = spec(k[\epsilon]/(\epsilon^2))$. Analogously, $F(k[\epsilon]/(\epsilon)^{10})$ is the set of deformations of X_0 to order 10.

We are going to stretch generality once more. Now assume we have any covariant functor $F : (Art) \longrightarrow (Sets)$. We will refer to elements of the set $F(R)$ just as 'deformations' in some very general sense. One can extend F to a functor $\widehat{F} : \widehat{(Art)} \longrightarrow (Sets)$ by putting

$$\widehat{F}(R) := \varprojlim F(R/\mathfrak{m}^k)$$

For R from (Art), clearly $F(R) = \widehat{F}(R)$. For a general non-artinian ring R, $\widehat{F}(R)$ consists of compatible systems of deformations $(\xi_k \in F(R/\mathfrak{m}^{k+1}))_{k\in\mathbb{N}}$ in the tower

$$\ldots \longrightarrow F(R/\mathfrak{m}^4) \longrightarrow F(R/\mathfrak{m}^3) \longrightarrow F(R/\mathfrak{m}^2) \longrightarrow F(R/\mathfrak{m})$$

We call such objects *formal deformations*[1]

Loosly speaking, a versal object was an object from which all other objects can be obtined by inducing. We introduce this concept here in the setting of functors.

Definition 9.1. A formal deformation $\hat{X} \in \widehat{F}(R)$, with R from $\widehat{(Art)}$ is called *versal* if for all $\psi : A' \longrightarrow A$ from (Art), all $\phi : R \longrightarrow A$ and all $X_{A'} \in F(A')$ with $\psi^*(X_{A'}) = (\phi^*(\widehat{X}))$, there exist a $\phi' : R \longrightarrow A'$, such that $X_{A'} = (\phi')^*(\widehat{X})$.

This looks cumbersome, but is a direct translation of **??**

Functors as generalised spaces

For R from $\widehat{(Art)}$ one has a canonical functor

$$h_R : (Art) \longrightarrow (Set)$$

[1] It should be stressed here that in many geometrically meaningful situations one starts with a functor already defined on a category of rings containing $\widehat{(Art)}$. In that case one should not confuse $F(R)$ and $\widehat{F}(R)$ for R not from (Art).

by putting $h_R(S) := Hom(R, S)$. In this way, spaces $Spec(R)$ correspond to certain functors h_R. Such functors are called *representable* functors, and a functor F of the form h_R is said to be represented by the ring R (or the space $Spec(R)$). One can try to extend geometrical notions from spaces, or rings to more general functors. It is useful to think of a functor as some kind of generalised space. A morphism $R' \longrightarrow R$ between rings induces for each S a map $Hom(R, S) \longrightarrow Hom(R', S)$ in a functorial way, so we obtain a transformation of functors:

$$h_R \longrightarrow h_{R'}$$

So the transformations of functors generalise maps between spaces. Note also that $h_R(k[\epsilon]) = Hom(R, k[\epsilon]) = (\mathfrak{m}/\mathfrak{m}^2)^*$ is the *Zariski tangent space* to $Spec(R)$. So in general we define the tangent space of a functor F to be just $T_F := F(k[\epsilon])$.

It is an astonishing fact that in this hopeless generality one can make a sensible definition of a *smooth transformation of functors*. To say that $f : F \longrightarrow G$ is a transformation of functors means that for any morphism $\phi : A' \longrightarrow A$ we get a canonical commutative diagram

$$\begin{tikzcd} F(A') \arrow[r, "f(A')"] \arrow[d, "F(\phi)"'] & G(A') \arrow[d, "G(\phi)"] \\ F(A) \arrow[r, "f(A)"] & G(A) \end{tikzcd}$$

and map $F(A') \longrightarrow G(A') \times_{G(A)} F(A) := \{(a, b) \in G(A') \times F(A) \mid G(\phi)(a) = f(A)(b)\}$

Definition 9.2. A transformation $f : F \longrightarrow G$ of functors is called *smooth*, if for all $A' \twoheadrightarrow A$ the canonical map $F(A') \longrightarrow G(A') \times_{G(A)} F(A)$ is *surjective.*

It makes sense to call this smoothness, because of the following theorem.

Theorem 9.3. $h_R \longrightarrow h_S$ *is smooth if and only if* $R = S[[x]]$

Given an $\widehat{X} \in \widehat{F}(R)$, one obtains a *transformation of functors*

$$PB(\widehat{X}) : h_R \longrightarrow F$$

obtained by *pulling-back*: To $\psi : R \longrightarrow A$ from $h_R(A)$ we associate $\psi^*(\widehat{X}) \in F(A)$. Looking at the diagrams defining smoothness and versality we see that:

Proposition 9.4. $\widehat{X}$ *is versal if and only if* $PB(\widehat{X})$ *is a smooth transformation.*

To construct a formal object having formally some property like this is based on the ideas of *small extensions* and *glueing*

Definition 9.5.

(1) An exact sequence

$$0 \longrightarrow V \longrightarrow R' \longrightarrow R \longrightarrow 0$$

is called a *small extension* if $\mathfrak{m}_{R'}V = 0$. In that case V aquires the structure of an k-vector space. Archetypical example is the sequence

$$0 \longrightarrow (\epsilon^k/\epsilon^{k+1}) \longrightarrow k[\epsilon]/(\epsilon^{k+1}) \longrightarrow k[\epsilon]/(\epsilon^k) \longrightarrow 0,$$

or more generally the sequence

$$0 \longrightarrow (\mathfrak{m}^k/\mathfrak{m}^{k+1}) \longrightarrow R/(\mathfrak{m}^{k+1}) \longrightarrow R/(\mathfrak{m}^k) \longrightarrow 0.$$

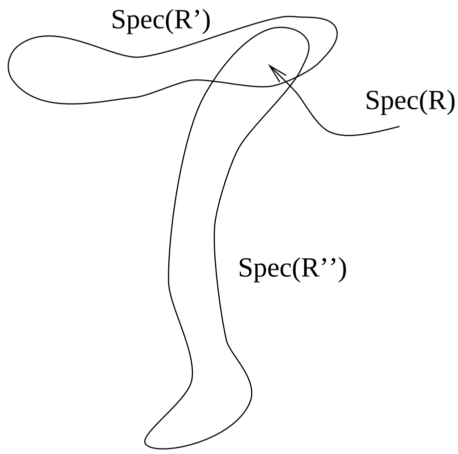
(2) If we have a diagram

$$\begin{tikzcd} R' \arrow[r, "\alpha"] & R \\ & R'' \arrow[u, "\beta"'] \end{tikzcd}$$

there exists a fibred sum-ring

$$R' \times_R R'' := \{(a,b) \in R' \times R'' \mid \ \alpha(a) = \beta(b)$$

with componentwise addition and multiplication. Geometrically, $Spec(R' \times_R R'')$ is obtained by glueing $Spec(R')$ and $Spec(R'')$ along $Spec(R)$.



Any such fibre-sum diagram

$$\begin{tikzcd} R' \arrow[r] & R \\ & R'' \arrow[u] \end{tikzcd}$$

induces for a functor F a canonical map

$$can : F(R' \times_R R'') \longrightarrow F(R') \times_{F(R)} F(R'')$$

Theorem 9.6. *Assume a functor F satisfies the following three conditions:*
(H1): For all diagrams of the form

$$\begin{tikzcd} R' \arrow[r] & k \\ & k[\epsilon] \arrow[u] \end{tikzcd}$$

the map can is a bijection.
(H2) For any diagram

$$\begin{tikzcd} R' \arrow[r] & R \\ & R'' \arrow[u] \end{tikzcd}$$

with $R' \longrightarrow R$ a small surjection, the map can is a surjection.
(H3) $\dim_k(T_F) < \infty$.

Then there exists a hull, or semi-universal formal object $\widetilde{X} \in \widetilde{F}(R)$ for F, that is, the map

$$PB(\widehat{X}) : h_R \longrightarrow F$$

is smooth and induces an isomorphism on tangent spaces.

Comment 9.7. Consider the glueing of two copies of $\mathbb{T}$. There is a diagram

$$\begin{tikzcd} k[\epsilon] \arrow[r] \arrow[d] & k \\ k[\epsilon] \times_k k[\epsilon] \arrow[r] & k[\epsilon] \arrow[u] \end{tikzcd}$$

The ring $k[\epsilon] \times_k k[\epsilon]$ is isomorphic to the ring $k[\epsilon, \epsilon']/(\epsilon^2, \epsilon\epsilon', (\epsilon')^2)$ via the map $(a + \epsilon b, a + \epsilon c) \mapsto (a+\epsilon b+\epsilon' c)$. Also, there is a map add : $k[\epsilon]\times_k k[\epsilon] \longrightarrow k[\epsilon]$ defined by $\text{add}(a+\epsilon b, a+\epsilon c) = (a+\epsilon(b+c))$. For a functor that satisifies (H1) one obtains a map

$$F(k[\epsilon])_k F(k[\epsilon]) \overset{\approx}{\longleftarrow} F(k[\epsilon] \times_k k[\epsilon]) \overset{F(add)}{\longrightarrow} F(k[\epsilon],)$$

that is, a map

$$T_F \times_k T_F \longrightarrow T_F$$

In this way, T_F aquires a natural structure of a k-vector space and condition (H3) makes sense.

Idea of the construction

(1) Choose a k-basis $\theta_1, \theta_2, \longrightarrow, \theta_\tau$ for the vector space T_F.

(2) Consider the formal power series ring $P := k[[T_1, T_2, \longrightarrow T_\tau]]$.

(3) We are going to define inductively ideals $I_n \subset P$ and objects $X_n \in F(S_n)$, $S_n := P/I_n$, such that

(4) $I_1 = \mathfrak{m}^2$, $S_1 = P/\mathfrak{m}^2 = k[\epsilon] \times_k k\epsilon \times_k \cdots k[\epsilon]$, $X_1 = \theta_1 \times \theta_2 \times \cdots \theta_\tau \in T_F \times \cdots T_F = F(S_1)$.

(5) Assume that I_n and X_n have been constructed. Put

$$\mathscr{L} = \{I \subset P \mid \mathfrak{m}I_n \subset J \subset I_n \& X_n \text{ lifts to } F(P/J)\}$$

This set of ideals is closed under intersection: given J_1 and J_2 in $\mathscr{L}$, we can form the following fibre-sum diagram:

$$\begin{tikzcd} P/J_1 \arrow[r, "\alpha"] & P/I_n \\ P/J_1 \cap J_2 \arrow[r] \arrow[u] & P/J_2 \arrow[u, "\beta"'] \end{tikzcd}$$

as $P/J_1 \cap J_2 = P/J_1 \times_{P/I_n} P/J_2$. Note that both α and β are small surjections, so by (H2) we can lift anything over P/J_1 and P/J_2 that restricts to the same over P/I_n to something over $P/J_1 \cap J_2$.

(6) Let I_{n+1} be the *minimal element* of $\mathscr{L}$ and let X_{n+1} be *any lift* of X_n over $S_{n+1} = P/I_{n+1}$.

(7) We can go on with this process for ever. In this way we find a compatible system of deformations, that is, an element of $\widehat{X} \in \widehat{F}(R)$ with $R = \overset{lim}{\longleftarrow} n\ (P/I_n)$!

It remains to check that the object so constructed is indeed versal in the above sense.

Theorem 9.8. *Let X_0 be any scheme over k. The deformation functor $Def(X_0)$, isomorphism classes of flat deformations of X_0, satisfies (H1) and (H2). Moreover, if* $\dim T(Def(X_0)) < \infty$, *then $Def(X_0)$ has a hull.*

Chapter 10

T^1 and T^2 for Singularities

Infinitesimal Deformations

In this part we work in the formal category. Let $f \in k[[x_1, \ldots, x_n]]$.

Definition 10.1. We say that f has an isolated singularity iff $\dim k[[x_1, \ldots, x_n]]/(\partial_1 f, \ldots, \partial_n f, f) < \infty$

We want to understand all flat deformations over the double point $\mathbb{T} = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$. We denote this set by $Def_X(\mathbb{T})$. $f + \epsilon g$ is flat exactly when we can lift the relations. But there is no non-trivial relation between one f, so we can take all $g \in k[[x_1, \ldots, x_n]]$. We have to divide out by infinitesimal automorphisms, which induce the identity for $\epsilon = 0$. These are given by:

$$x_j \mapsto x_j + \epsilon \alpha_j$$

for $j = 1, \ldots, n$, and $\alpha_j \in k[[x_1, \ldots, x_n]]$ can be arbitrary. This leads to the following deformation of f:

$$f(x_1 + \epsilon\alpha_1, \ldots x_n + \epsilon\alpha_n) = f(x) + \epsilon \sum \alpha_j \frac{\partial f}{\partial x_j}$$

(Recall that $\epsilon^2 = 0$.) So we see that the trivial deformations are generated by the the derivations Θ. We therefore get:

Theorem 10.2. *For a hypersurface singularity $X = V(f)$ we have*

$$T^1_X = Def_X(\mathbb{T}) \cong k[[x_1, \ldots, x_n]]/(\partial_1 f, \ldots, \partial_n f, f)$$

We therefore see that T^1_X is finite dimensional exactly when f has an isolated singularity.

We now consider more general X defined by $(f_1, \ldots, f_k) \subset k[[x]] = k[[x_1, \ldots, x_n]]$ and try to understand the flat deformations over the double point $T^1_X = Def_X(\mathbb{T})$ for those. We take the Ansatz:

$$(f_1 + \epsilon g_1, \ldots, f_k + \epsilon g_k)$$

For which, assuming it is flat, we can lift the relations. So let such a relation be given:

$$f_1 r_1 + \ldots + f_k r_k = 0$$

and let the lift be given by $r_i + \epsilon s_i$:

$$(f_1 + \epsilon g_1)(r_1 + \epsilon s_1) + \ldots + (f_k + \epsilon g_k)(r_k + \epsilon s_k) = 0$$

of course calculated modulo ϵ^2: Multiplying out we get: $\sum f_i r_i + \epsilon \sum (g_i r_i + f_i s_i)$. From this it follows that $\sum g_i r_i \in I := (f_1, \ldots, f_k)$, that is, zero in $\mathscr{O}_{X_S}$. It follows that the map:

$$I \longrightarrow \mathscr{O}_X \qquad f_i \mapsto g_i$$

is well defined! Therefore, to every flat deformation over $\mathbb{T}$ we can assign an element of $N_X := Hom(I, \mathscr{O}_X) = Hom(I/I^2, \mathscr{O}_X)$. This argument works the other way around to: every $\phi \in N_X$ gives rise to a flat deformation over $\mathbb{T}$, given by $f_1 + \epsilon\phi(f_1), \ldots, f_k + \epsilon\phi(f_k)$.

We still have to divide out by the automorphisms, which turn out, as in the hypersurface case to be generated by the derivations. In fact we have a map:

$$\Theta \longrightarrow N_X, \qquad \theta \mapsto (f_i \mapsto \theta(f_i) = g_i)$$

It is easy to see that this is well-defined. Let $r_1 f_1 + \ldots + f_k r_k = 0$. Applying θ and using the Leibnitz rule we get $\sum_i f_i \theta(r_i) + r_i \theta(f_i) = 0$, showing that $\sum r_i g_i \in I$. Therefore:

Theorem 10.3.

$$T^1_X = N_X/(Im(\Theta \to N_X)$$

Obstructions

We now turn our the the following question.

Problem 10.4. Suppose given a flat deformation of X over $\mathbb{T}$ given by

$$(f_1 + \epsilon g_1, \ldots, f_k + \epsilon g_k)$$

Does there exist a flat deformation of X over $Spec(\mathbb{C}[\epsilon]/(\epsilon^3))$ inducing the given flat deformation over $\mathbb{T}$.

To put it in another way, can the family be lifted to third order?

In general, the answer to this question is NO, but it is not so easy to give examples. What we want to do know, is to show that it can be solved from a computational point of view.

As by assumption our family is flat over $\mathbb{T}$ we know that for each relation $(r_1, \ldots, r_k)$ with $\sum f_i r_i = 0$ we can find $(s_1, \ldots, s_k)$ (depending of course on the relation) such that

$$(f_1 + \epsilon g_1)(r_1 + \epsilon s_1) + \ldots + (f_k + \epsilon g_k)(r_k + \epsilon s_k) = 0 \text{ modulo } \epsilon^2$$

There is no reason at all that this also holds modulo ϵ^3. We want to lift to third order, that is we want to find $h_1, \ldots, h_k$ such that our family to third order is given by:

$$(*) \qquad (f_1 + \epsilon g_1 + \epsilon^2 h_1, \ldots, (f_k + \epsilon g_k + \epsilon^2 h_k)$$

and is flat. So we have to find for all relations $(r_1, \ldots, r_k)$ between the f_i a lift $r_i + \epsilon s_i + \epsilon^2 t_i$ with

$$(**) \qquad \sum (f_i + \epsilon g_i + \epsilon^2 h_i)(r_i + \epsilon s_i + \epsilon^2 t_i) = 0 \text{ modulo } \epsilon^3$$

Lemma 10.5. *Given $h_1, \ldots, h_k$, the problem of lifting a relation $(r_1, \ldots, r_k)$ to third order is independent of the particular s_i chosen, as long as it satisfies equation $(*)$*

Proof. Let s'_i be another lift to second order satisfying $(*)$. Then

$$\sum (f_i + \epsilon g_i)(r_i + \epsilon s_i) = \sum ((f_i + \epsilon g_i)(r_i + \epsilon s'_i)$$

modulo ϵ^2. It follows that $\sum f_1(s_i - s_i') = 0$, that is $s_1 - s_1', \dots, s_k - s_k'$ is a relation between the f_i. As by assumption the $f_i + epsilon g_i$ is a flat deformation over $\mathbb{T}$, it follows that the relation can be lifted. Therefore there exis $u_1, \dots, u_k$ such that

$$\sum g_i(s_i - s_i') + \sum f_i u_i = 0$$

Given $h_1, \dots, h_k$, suppose that we can lift the relation with s_i. Then we can find $t_1, \dots, t_k$ such that $(**)$ holds, or by looking at ϵ^2-term:

$$\sum (h_i r_i + g_i s_i + t_i h_i) = 0.$$

The question is whether we can find t_i' such that

$$\sum (h_i r_i + g_i s_i' + t_i' h_i) = 0.$$

By subtracting this is equivalent to finding t_i' with

$$\sum f_i(s_i - s_i') + \sum g_i(t_i - t_i') = 0$$

This is possible by defining $t_i' = t_i - u_i$ for $i = 1, \dots, k$. $\square$

Coming back to the lifting question, we want the following equation to hold:

$$\sum (r_i h_i + g_i s_i + t_i f_i) = 0$$

We can also read this as:

(†) $$\sum (r_i h_i + g_i s_i) = 0 \in \mathscr{O}_X$$

as then it is possible to find the t_i as above. Putting $\mathscr{R}$ to be te module of relations between the $f_1, \dots, f_k$ we consider the map

$$ob(\underline{g}) : \mathscr{R} \longrightarrow \mathscr{O}_X : \underline{r} = (r_1, \dots, r_k) \mapsto \sum g_i s_i$$

the lemma above can be reformulated to say that the map is well-defined.

Lemma 10.6. *Let $\mathscr{R}_0$ be the submodule of $\mathscr{R}$ generated by relations of the type:*

$$(0, \dots, f_j, 0 \dots, 0, -f_i, 0, \dots, 0)$$

where it is supposed that f_j is on the i'th spot, and $-f_i$ is on the j'th spot.

The proof is left as an exercise. The map $ob(\underline{g})$ therefore descends down to a map:

$$ob(\underline{g}) : \mathscr{R}/\mathscr{R}_0 \longrightarrow \mathscr{O}_X$$

The he question of lifting the family to a flat family of third order can be reformulated as saying that the map is (†) is of special type. We need to find $h_1, \dots, h_k$ such that the map $ob(g\underline{g}$ is of type:

$$(r_1, \dots, r_k) \mapsto \sum h_i r_i$$

This motivates the following definition

Definition 10.7. Consider a presentation of the ideal $I = (f_1, \dots, f_k)$ as $k[[x]]$-modules:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{F} \stackrel{(f_1,\dots,f_k)}{\longrightarrow} I$$

This induces a map:

$$Hom(\mathscr{F}, \mathscr{O}_X) \longrightarrow Hom(\mathscr{R}/\mathscr{R}_0, \mathscr{O}_X)$$
$$h_i \mapsto (r_i \mapsto \sum h_i r_i)$$

Then we define:

$$T_X^2 = Hom(\mathscr{R}/\mathscr{R}_0, \mathscr{O}_X)/Hom(\mathscr{F}, \mathscr{O}_X)$$

We showed the following Theorem:

Theorem 10.8. *The deformation $\underline{g}$ over $\mathbb{T}$ can be lifted to third order if and only if the element:*

$$ob(\underline{g}) \in T_X^2$$

is zero.

Having lifted to third order, the question is whether one can lift to fourth order. It turns out that one gets an obstruction element associated to this situation again. This obstruction element is in T_X^2 again! For this, and more, we refer to the exercises.

Exercise 10.9. Prove 7.2

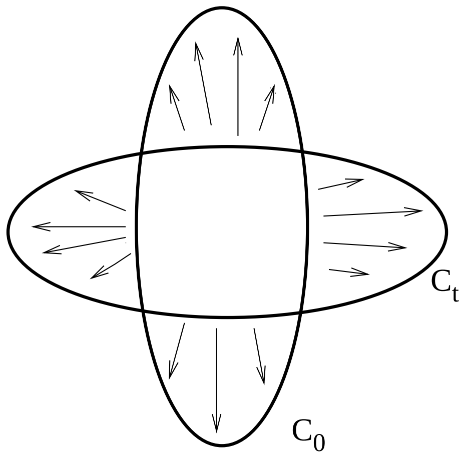
Chapter 11

Curves on Surfaces I

Let F be a smooth compact complex surface. We will consider families of curves

$$C_s \subset F \times S$$

Suppose $0 \in S$. The following picture



shows that a tangent vector to S "is" a vectorfield on C_0. This vectorfield is well–defined only up to tangent vectorfields, giving an element in the normal bundle $\mathscr{N}$. The normal sheaf sits in the following exact sequence

$$0 \longrightarrow \Theta_{C_0} \longrightarrow \Theta_{F|C_0} \longrightarrow \mathscr{N}_{C_0/F} \longrightarrow 0$$

and is defined by

$$\mathscr{N}_{C_0/F} = \mathscr{H}om(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_{C_0})$$

Here $\mathscr{J}$ is the ideal sheaf of C_0. Locally, the ideal sheaf is generated by an element $f_0(x, y)$, and the family is given locally by:

$$f(x, y) = f_0(x, y) + sf_1(x, y) + O(s^2).$$

Both constructions give a map, called the "characteristic map":

$$\rho : T_0 S \longrightarrow H^0(C, \mathscr{N}_{C_0/F})$$

We look in more detail to the case:

- C_0 is a curve of degree d in $\mathbb{P}^2$.

- S is the linear system of all curves of degree d, so S is in fact equal to $\mathbb{P}^{\binom{d+2}{2}-1}$.

The normal bundle sequence then looks like:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_{\mathbb{P}^2} & \longrightarrow & \mathscr{O}_{\mathbb{P}^2}(C_0) & \longrightarrow & \mathscr{N}_{C_0/\mathbb{P}^2} & \longrightarrow 0 \\ & & 1 & \longmapsto & \frac{1}{f} & \longmapsto & (f \mapsto 1) & \end{array}$$

By taking global sections we get an isomorphism:

$$H^0(\mathscr{O}_{\mathbb{P}^2}(C_0))/\mathbb{C} \approx H^0(\mathscr{N}_{C_0/\mathbb{P}^2})$$

Therefore $S = \mathbb{P}^{\binom{d+2}{2}-1}$ is a universal object for families of degree d. Curves can be given by equations (multi-index notation)

$$\sum a_i x^i$$

The a_i give the (homogeneous) coordinates for the base space S.

We now impose singularities, that is, we consider curves with fixed types of singularities, say of type $\underline{T} = (T_1, \ldots, T_L)$

Definition 11.1.

$$\Sigma_d^{\underline{T}} = \{f_d(x,y,z) \mid f_d = 0 \text{ has } k \text{ singular points of type } T_1, \ldots, T_k\}/\mathbb{C}^*$$

In general this space will not be linear, as we will see later, **??**

Definition 11.2. For a curve $C \in \Sigma_d^{\underline{T}}$. Let the singular points of C be $p_1, \ldots, p_k$. We define the sheaf $\mathscr{N}_C'$ by the following exact sequence:

$$0 \longrightarrow \mathscr{N}_C' \longrightarrow \mathscr{N}_C \longrightarrow \oplus_{i=1}^k T^1_{(C,p_i)} \longrightarrow 0$$

Theorem 11.3 (Wahl). *The tangent space to $\Sigma_d^{\underline{T}}$ in C is $H^0(\mathscr{N}_C')$. The "obstructions" lie in $H^1(\mathscr{N}_C')$. The formal completion of $\Sigma_d^{\underline{T}}$ at C is the fibre of a map $ob : H^0(\mathscr{N}_C') \longrightarrow H^1(\mathscr{N}_C')$.*

Remark 11.4. It follows that from $H^1(\mathscr{N}_C') = 0$, that $\Sigma_d^{\underline{T}}$ is smooth. and we have an exact sequence:

$$0 \longrightarrow H^0(\mathscr{N}_C') \longrightarrow H^(\mathscr{N}_C) \longrightarrow \oplus_{i=1}^k T^1_{(C,p_i)} \longrightarrow 0$$

We now come to a famous example of a $\Sigma_d^{\underline{T}}$ which is not smooth:

Example 11.5 (Luengo). $\Sigma_9^{A_{35}}$ is not smooth.

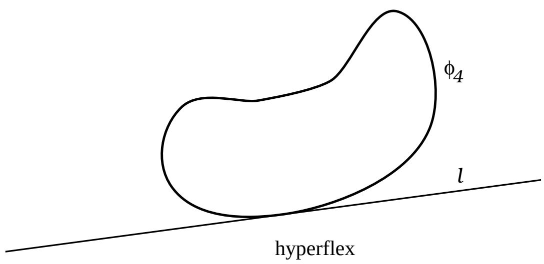
The singular point of $\Sigma_9^{A_{35}}$ is the curve C giving by the equation:

$$f(x,y,z) = x^9 + y(xy^3 + z^4)^2$$

It has indeed an A_{35}–singularity, at $(0:1:0)$. Look at the affine chart $y = 1$, put $\xi = x + z^4$, and we get equation

$$\xi^2 + (\xi - z^4)^9 = \xi^2 + z^{36} + hot = 0$$

so that we indeed see that it has an A_{35}–singularity. The form of the equation is:



$l^9 + \phi_4^2 \cdot n = 0$. The hyperflex intersects the curve ϕ_4 with multiplicity four. Curves of this form form a 16–dimensional family. (Why?) The expected dimension of $\Sigma_9^{A_{35}}$ is $\binom{9+2}{2} - 1 - 35 = 19$.

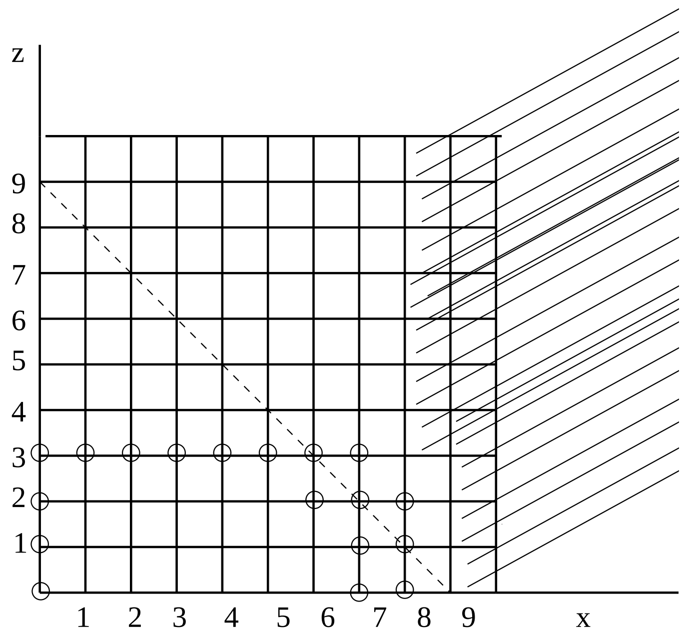
Theorem 11.6. *The special curves form the singular locus of the* $\Sigma_9^{A_{35}}$.

We want to understand the following exact sequence:

$$0 \to H^0(\mathscr{N}') \to H^0(\mathscr{N}) \to T^1_{C,p} \to H^1(\mathscr{N}') \to 0$$

Here $H^0(\mathscr{N}) \cong \mathbb{C}^{54}$ $(54 = \binom{11}{2} - 1)$, the space of monomials in x, z of degree ≤ 9. p is the singular point of C, and has therefore dimension 35, because C has an A_{35}–singularity.

$$\begin{array}{rcl} f & = & x^9 + (x + z^4)^2 \\ \partial_x & : & 9x^8 + 2(x + z^4) \\ \partial_z & : & 8z^3(x + z^4) \end{array}$$



Put $J = (\partial_x(f), \partial_z(f)$. We have $x^9 = \frac{4\partial_x(f) - \partial_z(f)}{36} \in J$. Similarly, $x^8z^3 \in J$. Furthermore, every monomial with a z^4 in it, can be reduced modulo J to something in x (look at $\partial_x(f)$).

A basis of T^1 is represented by $x^i z^j$, $i < 9, j < 4$, $(i, j) \neq (8, 3)$. The map $H^0(\mathscr{N}) \to T^1_(C, p)$ is given by the "picture". Obviously, all elements of T^1 which can be represented by monomials of degree ≤ 9 are in the image of the map. What about x^8z^2 and x^7z^3. Now

$$x^8 z^2 \equiv -\frac{2}{9} z^2 (x + z^4) \in Im(H^0(\mathscr{N}) \to T^1_p)$$

but $x^7z^3 \notin Im(H^0(\mathscr{N}) \to T^1_p)$. (Check this). We conclude:

Lemma 11.7. $H^1(\mathscr{N}) = [x^7z^3] \cong \mathbb{C}$, $H^0(\mathscr{N}') = 20$.

So we have four transverse directions to the special family considered above. Thus we get four interesting elements of $H^0(\mathcal{N}')$ which keep the A_{35} singularity to first order, but not to higher order. Those elements are given by:

$$x^9 + (x + z^4)^2 + 2(x + z^4) \cdot (\epsilon_{60}x^5 + \epsilon_{51}x^4z + \epsilon_{42}x^3z^2 + \epsilon_[3x^2z^3)$$

The fact that this deformation keeps to first order the A_{35}–singularity, can be seen by simply "completing the square":

$$((*)) \qquad x9 + (x + z^4 + \epsilon_{60}x^5 + \epsilon_{51}x^4z + \epsilon_{42}x^3z^2 + \epsilon_{33}x^2z^3)^2$$

This expression is of degree 10, hence does not globalise. By using coordinate transformation to second order, one can remove some monomials of degree 10, like x^8z^2. But x^7z^3 cannot be removed. The coefficient of $(*)$ of x^7z^3 is:

$$\epsilon_{60} \cdot \epsilon_{33} + \epsilon_{51}\epsilon_{42}$$

This gives the quadratic part of the equation for Σ_9^{35} at the point p, so in particular it shows that Σ_9^{35} is not smooth at that point. If one tries to find the higher order equations for Σ_9^{35} one runs into terrible computations, so here we better stop.

Chapter 12

Cohomology

To compute the simplicial homology of, say, a smooth compact manifold one can start by triangulating the manifold. One has then a (finite) number of vertices, edges, 2-faces, ..., top dimensional faces with boundary maps. The homology with values in an abelian group G is the homology of the complex $(K_., \partial)$ with K_i the free G-module with the i-faces as basis and the differential ∂ the boundary map extended by linearity.

Cohomology is computed with the dual complex. Concretely this means that a cochain f assigns to each vertex p_i a group element f_i, to an edge p_{ij} connecting the vertices p_i and p_j an element f_{ij}, etc. The differential is given by $(d\,f)(p_{ij}) = f(\partial\,p_{ij}) = f_i - f_j$, $(d\,f)(p_{ijk}) = f_{ij} + f_{jk} + f_{ki}$, etc. A cochain f is called closed if $d\,f = 0$ and exact if f is of the form $d\,\omega$.

Similar formulas appear in the definition of Čech cohomology. Consider a space X and a covering $\{U_i\}$. One has the following correspondance:

covering		triangulations
U_i	$\leftrightarrow$	vertices
$U_i \cap U_j$	$\leftrightarrow$	edges
$U_i \cap U_j \cap U_k$	$\leftrightarrow$	2-faces
$\cdots$		$\cdots$

Given a sheaf of rings $\mathscr{F}$ on X we can now associate to each U_i a section $f_j \in \mathscr{F}(U_i)$, to an intersection $U_i \cap U_j$ a section $f_{ij} \in \mathscr{F}(U_i \cap U_j)$ etc. We obtain the same formulas for the differential as above. A 0-cochain F is again closed if $f_i - f_j = 0$ on $U_i \cap U_j$.

We get a complex

$$\begin{array}{ccccccc} C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots \\ \oplus\mathscr{F}(U_i) & \xrightarrow{d} & \oplus\mathscr{F}(U_i \cap U_j) & \xrightarrow{d} & \oplus\mathscr{F}(U_i \cap U_j \cap U_k) & \xrightarrow{d} & \cdots \end{array}$$

where the first map d is defined by $d\,f_{|U_i \cap U_j} = f_i - f_j$, the second by $d\,f_{|U_i \cap U_j \cap U_k} = f_{ij} + f_{jk} + f_{ki}$ etc. One checks that $d^2 = 0$

Example 12.1. One has that $H^0(X, \mathscr{F})$ are the global sections of $\mathscr{F}$. In particular for $\mathscr{F} = \mathscr{O}_X$ the vector space $H^0(\mathscr{O}_X)$ is the space of global functions on X.

Example 12.2. $H^1(\mathscr{O}_X^*)$: this group classifies line bundles on X because it is the space of transition functions modulo equivalence. In this case we write the sections multiplicative. As notational convenience we set $\varphi_{ji} = (\varphi_{ij})^{-1}$. Then the cocycle condition $\varphi_{ij}\varphi_{jk}\varphi_{ki} = 1$ for 1-cochains translates into $\varphi_{ik} = \varphi_{ij}\varphi_{jk}\varphi_{ij}\varphi_{jk}$ whereas one obtains isomorphic bundles from transition functions φ'_{ij} which can be written as $\varphi'_{ij} = \frac{f_i}{f_j}\varphi_{ij}$ for a system of functions $f_i \in \Gamma(U_i, \mathscr{O}_X^*)$.

The tangent sheaf

If X is an analytic manifold then the tangent sheaf Θ_X is the sheaf of analytic sections of a vector bundle, namely the tangent bundle. Given a covering $\{U_i\}$ of X with small enough open sets one has a covering $\{U_i \times \mathbb{C}^n\}$ of the tangent bundle TX. If $(z_1, \ldots, z_n)$ are local coordinates on U_i and $(z'_1, \ldots, z'_n)$ on U_j then the transition function φ_{U_i,U_j} is given by the matrix

$$\begin{pmatrix} \frac{\partial z_1}{\partial z'_1}(p) & \cdots & \frac{\partial z_n}{\partial z'_1}(p) \\ \vdots & & \vdots \\ \frac{\partial z_1}{\partial z'_n}(p) & \cdots & \frac{\partial z_n}{\partial z'_n}(p) \end{pmatrix}$$

The group $H^0(\Theta)$ consists of the global vector fields on X. This is the associated Lie-algbra of the automorphism group $\mathrm{Aut}(X)$.

Example 12.3. If $X = \mathbb{P}^1$ then $\mathrm{Aut}(X) = \mathrm{PGl}(2, \mathbb{C})$ and $H^0(\Theta_X) \cong \mathrm{Sl}(2, \mathbb{C})$.

The number of zeroes of a section of a rank n vector bundle $\mathscr{F}$ is $c_n(\mathscr{F})$. In particular $c_n(\Theta)$ equals e, the Euler Number.

Now we come to $H^1(\Theta)$. The complex manifold structure on X is determined by the coordinate changes between local coordinates on open sets of an open covering. Let $X = \cup_{i=1}^n U_i$ where each U_i is isomorphic to the unit disc with coordinates z_i (thi is a vector). The transition functions $z_i := F_{ij} z_j$ are holomorphic on the domain of definition. Of course, whenever defined, we have

$$F_{ik} = F_{ij} F_{jk} \ .$$

Now we take a one parameter infinitesimal deformation of X, i.e. we consider a manifold $X_{\mathbb{T}}$ over

$$\mathbb{T} := \mathit{Spec}(\mathbb{C}[\varepsilon])$$

where $\varepsilon^2 = 0$. The idea (due to Kodaira and Spencer) is to take a covering:

$$X_{\mathbb{T}} = \cup_{i=1}^n (U_i \times \mathbb{T})$$

We perturb this situation, i.e. we look at transition functions $\mathbb{F}_{ij}$ which now are depending on z_j and ε, and such that for $\varepsilon = 0$ we get back our F_{ij}. We have the condition that on $U_i \cap U_j \cap U_k$:

$$\mathbb{F}_{ik}(z_k, t) = \mathbb{F}_{ij}(\mathbb{F}_{jk}(z_k, \varepsilon), \varepsilon) \ .$$

Writing $\mathbb{F}_{ij} = F_{ij} + \varepsilon G_{ij}$ we can consider G_{ij} as a vector field on $U_i \cap U_j$, explicitly:

$$\theta_{ij} = \sum_{\alpha=1}^{n} G_{ij}^{(\alpha)} \frac{\partial}{\partial z_i^{(\alpha)}} \ .$$

The equation

$$F_{ij}(F_{jk} + \varepsilon G_{jk}) + \varepsilon G_{ij}(F_{jk}) = F_{ik} + \varepsilon G_{ik}$$

yields using the chain rule the equation between vector fields

$$\theta_{ij} + \theta_{jk} = \theta_{ik}$$

because

$$F_{ij}(F_{jk} + \varepsilon G_{jk}) = F_{ij}(F_{jk}) + \varepsilon \frac{\partial F_{ij}}{\partial z_j} G_{jk}$$

and $\frac{\partial F_{ij}}{\partial z_j} = \frac{\partial z_i}{\partial z_j}$ is just the Jacobian, which as we saw gives the transition functions on the tangent bundle.

We conclude that our collection of vector field θ_{ij} satisfy the cocycle condition. It is boring to check that thie resulting element in first Cech cohomology group $H^1(X, \Theta_X)$ is independent of the choices made. On the other hand, given a cocyle $g_{ij}\frac{\partial}{\partial z_i}$ one defines a deformation over $\mathbb{T}$ by giving its transition functions $F_{ij} = f_{ij} + \epsilon g_{ij}$This deformation turns out to be trivial exactly when we have a coboundary.

Theorem 12.4. *The infinitesimal deformations of a complex manifold X over $\mathbb{T}$ are classified by $H^1(X, \Theta_X)$*

Chapter 13

Curves on Surfaces II

Take a family of curves on a projective surface F. Take on $F \times S$ an effective Cartier divisor which is flat over S.

Theorem 13.1. *A 1–parameter family of curves C_t is flat*

$\Longleftrightarrow$

The degree of C_t and the (arithmetic) genus C_t is constant

More generally one has, that a family X_t in projective space is flat, if and only if the Hilbert polynomial is constant in t.

Proof. The Hilbert polynomial

$$\dim H^0(X_t, \mathscr{O}_{X_t}(m)) =: h_t(m)$$

is a polynomial in m for $m >> 0$. Flatness means that t is a nonzero–divisor, that is, we have an exact sequence:

$$0 \longrightarrow \mathscr{O}_{X_T} \xrightarrow{\cdot t} \mathscr{O}_{X_T} \longrightarrow \mathscr{O}_{X_0} \longrightarrow 0$$

For $m >> 0$ one has that $H^1(\mathscr{O}_{X_T}(m)) = 0$. We get the following exact sequence:

$$0 \longrightarrow H^0(\mathscr{O}_{X_T}(m)) \xrightarrow{\cdot t} H^0(\mathscr{O}_{X_T}(m)) \longrightarrow H^0(\mathscr{O}_{X_0}(m)) \longrightarrow 0$$

So one sees that the rank of $\longrightarrow H^0(\mathscr{O}_{X_T}(m))$ as $\mathbb{C}\{t\}$–module is the same as the vectorspace dimension of $H^0(\mathscr{O}_{X_0}(m))$. □

Problem 13.2. Does ther exist a universal family?

Answer is YES, (Grothendieck), and is called the Hilbert scheme.

How to give the Hilbert scheme coordinates. Well, take a curve $C \subset F \subset \mathbb{P}^n$. We have *fixed* the Hilbert polynomial P. Take $m >> 0$, and look at all polynomilas $\mathbb{C}[X]/I_F$ which vanish on C and has degree m, that is (H is the hyperplane–divisor):

$$H^0(F, \mathscr{O}_F(-C + mH))$$

The dimenions of this vector–space is independent of C, but only depends on the fixed Hilbert polynomial P. We get the linear subspace:

$$H^0(F, \mathscr{O}_F(-C + mH)) \subset H^0(F,)_F(mH))$$

This subspace characterizes C. It gives therefore a point in a Graßmannian. This leads to the Hilbert scheme $Hilb^P(F) =: \Sigma^P$. (Recall that P is the Hilbert polynomial.)

Theorem 13.3. *Let be given a curve C and let $[C] \in \Sigma^P$ be the corresponding point in the Hilbert scheme. We have a characteristic map:*

$$\rho : T_{[C]}\Sigma^P \longrightarrow H^0(C, \mathscr{N}_C)$$

The map ρ is an isomorphism. (This is more or less a tautology!) The "obstructions lie in $H^1(C, \mathscr{N}_C)$". In particular, if $H^1(C, \mathscr{N}_C) = 0$, then Σ^P is smooth in $[C]$.

Proof. Let $C_A \subset F \times \mathrm{Spec}(A)$, A artinian. Take an exact sequence:

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0.$$

We suppose that I is a one–dimensional vector–space, with generator η. An example is $A = \mathbb{C}[\epsilon]/(\epsilon^n), A' = \mathbb{C}[\epsilon]/(\epsilon^{n+1}), \eta = \epsilon^n$.

We have open sets:

U_i on F; $F_i = 0$ local equation of C_A defined on $U_i \times \mathrm{Spec}(A)$.

U_j on F; $F_j = G_{ij}F_j$ with $G_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}^*)$.

Take an arbitrary lift F'_i, G'_{ij}. Then

$$F'_i - G'_{ij}F'_j = \eta \cdot h_{ij} \qquad ((*))$$

We want to show that the h_{ij} naturally defines an element in $H^1(\mathscr{N}_C)$.

By restricting to the zero fibre, we also have the f_i, local equation for our original curve. It is defined over $\mathbb{C} = A/\mathfrak{m}A$. The normal sheaf of C locally on U_i is generated by:

$$f_1 \mapsto 1$$

The condition that $f_i \mapsto h_{ij}$ on $U_i \cap U_j$ is a cocycle is as follows: Look at $U_i \cap U_j \cap U_k$, and compute the coboundary:

$$f_i \mapsto h_{ij} - h_{ik} + g_{ij}h_{jk}$$

where $g_{ij} = \frac{f_i}{f_j}$. Using the definition of h_{ij}, see $(*)$, we get:

$$\eta(h_{ij} - h_{ik} + G'_{ij}h_{jk}) = F'_i - G'_{ij}F'_j - F'_i + G'_{ik}F_k + G'_{ij}(F'_j - G'_{jk}F'_k) = (G'_{ik} - G'_{ij}G'_{jk})F'_k$$

As the G'_{ij} are lifts of the G_{ij}, the term $(G'_{ik} - G'_{ij}G'_{jk})$ is divisible by η. We divde by η, and calculate modulo $\mathfrak{m}_A$ to get the following equation:

$$h_{ij} - h_{ik} + g_{ij}h_{jk} = \frac{G'_{ik} - G'_{ij}G'_{jk}}{\eta \cdot f_k} \qquad (\,**)$$

The right hand side is zero in $\mathscr{O}_C$ showing that the h_{ij} indeed is a cocyle. It therefore defines an element in $H^1(\mathscr{N}$. It remains to show, that if the h_{ij} is a coboundary, (that is the zero element in $H^1(\mathscr{N}$, that then the family can be lifted. So, suppose that h_{ij} is a coboundary. Then we have:

$$f_i \mapsto k_i \text{ on } U_i$$

whose coboundary should give our h_{ij}. This coboundary is given by:

$$f_i \mapsto \frac{f_i}{f_j}k_j - k_i \text{ on } \mathscr{O}_{C|U_i \cap U_j}$$

This should be equal to h_{ij} but of course, only in $\mathscr{O}_C$. We can therefore find elements l_{ij} with:

$$h_{ij} = \frac{f_i}{f_j}k_j - k_i + l_{ij}f_j$$

Now one calculates directly that:

$$(F'_i + \eta k_i) - (G'_{ij} + \eta l_{ij})(F'_j + \eta k_j) = 0$$

so that we found a lift of our given family over $\mathrm{Spec}(A')$. □

Therefore, if $H^1(\mathscr{N}_C = 0$, the Hilber schem Σ^P is smooth. A weaker condition can be formulated. Consider the exact sequence:

$$0 \longrightarrow \mathscr{O}_F \longrightarrow)_f(C)\ lra\mathscr{N}_C \longrightarrow 0$$

It leads to the long exact cohomology sequence:

$$\longrightarrow H^1(\mathscr{O}_F(C)) \longrightarrow H^1(\mathscr{N}_C) \xrightarrow{\delta} H^2(\mathscr{O}_F$$

Theorem 13.4. *If δ is injective then Σ^P is smooth in $[C]$.*

Proof. The coboundary δ has to be computed:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_k & \longrightarrow & B_k & \longrightarrow & C_k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_{k+1} & \longrightarrow & B_{k+1} & \longrightarrow & C_{k+1} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_F & \longrightarrow & \mathscr{O}_F(C) & \longrightarrow & \mathscr{N}_C & \longrightarrow & 0 \\ & & 1 & \mapsto 1 = \frac{f_i}{f_i} & & & & & \\ & & & & \frac{1}{f_i} & \mapsto & (f_i \mapsto 1) & & \end{array}$$

Divide $(**)$ by f_i (Recall that $f_k = \frac{f_i}{g_{ik}}$ etc.)

$$\frac{h_{ij}}{f_i} = -\frac{h_{ik}}{f_i} + \frac{h_{jk}}{f_i} = \frac{1 - G'_{ij} G'_{jk} G'^{-1}_{ik}}{\eta}$$

This is a coboundary:

$$\sigma_{ijk} = \frac{1 - G'_{ij} G'_{jk} G'^{-1}_{ik}}{\eta} \in H^2(\mathscr{O}_F).$$

The element σ_{ijk} is the obstruction to lift G_{ij} from $\mathscr{O}_F \otimes A)^*$ to $\mathscr{O}_F \otimes A')^*$. Over $\mathbb{C}$ the map

$$\mathscr{O}_F \otimes A')^* \longrightarrow \mathscr{O}_F \otimes A)^*$$

splits. as $\mathscr{O}_F \otimes A)^* \cong \mathscr{O}_F^*(1 + \mathscr{O}_F \otimes \mathfrak{m}_A)$. We have the exponential map:

$$exp : (sO_F \otimes \mathfrak{m}_A)_+ \cong 1 + \mathscr{O}_F \otimes \mathfrak{m}_A$$

and the sequence $A' \longrightarrow A$ splits additively. $\square$

By a curve on a smooth surface we mean an effective (Cartier) divisor. A Cartier divisor is a global section of $\mathscr{K}^*/\mathscr{O}^*$: $\mathscr{K}$ is the function field of F: $\mathscr{K}$ is a constant sheaf. A Cartier divisor can be given by a local equation. Recall that

$$D_1 \overset{lin}{\sim} D_4\ if\ D_1 - D_2 = (f); f \in \mathscr{K}^*$$

that, is D_1 and D_2 are in the same linear system.

$C_1 \sim C_2$: if C_1 and C_2 are in the same flat family of curves. C_1 and C_2 are called algebraically equivalent.

This is not an equivalence relation, so make it into one by taking the transitive hull. From the sequence:

$$0 \longrightarrow \mathscr{O}^* \longrightarrow \mathscr{K}^* \longrightarrow \mathscr{K}^*/\mathscr{O}^* \longrightarrow 0$$

we get

$$0 \longrightarrow H^0(\mathscr{K}^*)/\mathbb{C}^* \longrightarrow H^0(\mathscr{K}^*/\mathscr{O}^*) \longrightarrow H^1(\mathscr{O}^*) \longrightarrow 0$$

$H^1(\mathscr{O}^*)$ is called the Picard group. It has a lots of components. $Pic^0(F)$ is called the Picard variety: this all all elements in Pic which are algebraically equivalent to zero modulo linear equivalence.

Severi conjectured that

$$\dim Pic^0 := \dim H^1(\mathscr{O}).$$

This is true for over $\mathbb{C}$, bot is wrong in characteristic p.

Proof. The exponential sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O} \stackrel{exp}{\longrightarrow} \mathscr{O}^* \longrightarrow 0$$

gives rise to:

$$H^1(\mathbb{Z}) \longrightarrow H^1(\mathscr{O}) \longrightarrow H^1(\mathscr{O}^*) \longrightarrow H^2(\mathbb{Z})$$

(So this is a transcendental proof, and therefore not valid for characteristic p. □

Furthermore we have:

$$0 \longrightarrow \mathscr{O}_F \longrightarrow \mathscr{O}_F(C) \longrightarrow \mathscr{N}_C \longrightarrow 0$$

whose cohomology gives:

$$0 \longrightarrow H^0(\mathscr{O}_F(C))/\mathbb{C}^* \longrightarrow H^0(\mathscr{N}_C) \longrightarrow H^1(\mathscr{O}_F)H^1(\mathscr{O}_F(C))$$

There are a lot of curves $\deg(C) >> 0$ wiht $H^1(\mathscr{O}_F(C)) = 0$. Now

$H^0(\mathscr{N}_C)$: Zariski tangent space to the Hilbert scheme,

$H^0(\mathscr{O}_F(C))$: Tangent space to the linear system,

and of course $H^1(\mathscr{O}_F)$ "does not depend on C". Therefore:

Theorem 13.5. *If the Hilbert scheme is smooth in $[C]$, the the dimension of divisors algebraically equivalent to zero, modulo linear equivalence, can be computed from the tangent spaces.*

This gives q as dimension.

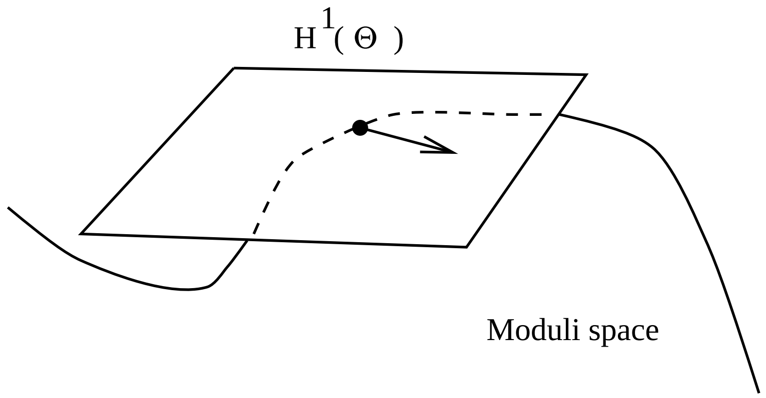
Chapter 14

On how many parameters does a surface depend?

Let X be a complex manifold, with a covering by small opens U_i and with transition functions F_{ij}. One gets all the complex structures on X by deforming the F_{ij}. What are the first order perturbations of the F_{ij}? Because $f(a+\epsilon g) = f(a) + \epsilon f'(a).g + \ldots$, perturbing $F_{i,j}$ to $F_{ij} + \epsilon G_{ij}$ leads to a Cech 1-cocycle in Θ_X, the tangent sheaf of X. Hence, the first order perturbations correspond to

$$H^1(\Theta_X)$$

which therefore is the tangent space to the set of complex structures.



How to compute $H^1(\Theta)$ for a given complex variety? For a Riemann surface X of genus g this was simple. If $g \geq 2$ one has $H^0(\Theta) = 0$, so it follows from Riemann-Roch that $\dim H^1(\Theta_X) = 3g - 3$. Now suppose X is a complex surface. Of course we always can compute $\chi(\Theta)$ from Riemann-Roch.

$$\chi(\Theta) = \dim H^0(\Theta) - \dim H^1(\Theta) + \dim H^2(\Theta)$$

Riemann Roch for a line bundle $\mathscr{O}(D)$ on a surface reads:

$$\chi(\mathscr{O}_D) = \frac{1}{2}D(D-K) + \chi(\mathscr{O})$$

If $V = \mathscr{O}(D_1) \oplus \mathscr{O}(D_2)$ is a decomposed rank two bundle, then $h^i(V) = h^i(\mathscr{O}(D_1)) + h^i(\mathscr{O}(D_2))$, so we find

$$\chi(V) = \frac{1}{2}(D_1^2 + D_2^2 - (D_1 + D_2)K) + 2\chi(\mathscr{O})$$

Now $c_1(V) = D_1 + D_2$, $c_1^2(V) = (D_1 + D_2)^2 = D_1^2 + D_2^2 + 2D_1 \cdot D_2$, $c_2(V) = D_1 \cdots D_2$, so we can write

$$\chi(V) = \frac{1}{2}(c_1(V)^2 - 2c_2(V) - c_1(V)K) + 2\chi(\mathscr{O})$$

which now makes sense and in fact is true for any rank two bundle on a surface. Let us apply it to $V = \Theta$: $c_1(\Theta) = -K$, $c_2(V) = e$. By Noether's formula, $\chi(\mathscr{O}) = \frac{1}{12}(e + K^2)$. Plugging in everything, we get

$$\chi(\Theta) = \frac{1}{6}(7K^2 - 5e)$$

Let us take a look at some examples: if X is a torus, or a K3-surface, $\Omega^1 \approx \Theta$, hence $H^1(\Theta) = H^1(\Omega^1)$ has the Hodge number $h^{1,1}$ as dimension. We find:

	Abelian	$K3$
$H^0(\Theta)$	2	0
$H^1(\Theta)$	4	20
$H^2(\Theta)$	2	0

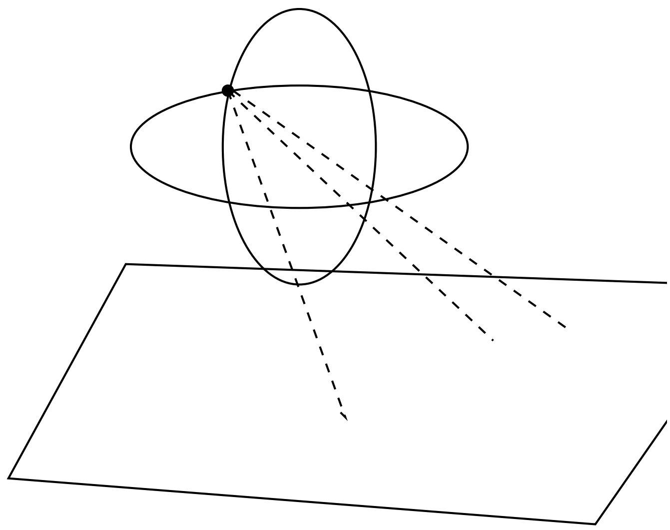
Indeed, an abelian surface $A = \mathbb{C}^2/\Lambda$ depends on the choice of a lattice with four base vectors in $\mathbb{C}^2$, $= 2 \times 4$ parameters, but there acts a group $Mat(2 \times 2)$, reducing the number to 4.

Now let us take a look at K3's: the simplest example is a quartic in $\mathbb{P}^3$. A polynomial $F(X, Y, Z, T)$ homogeneous of degree 4 has 35 coefficients. The group $GL(4)$ of 4×4 matrices acts, so we are left with $35 - 16 = 19$ parameters. But:

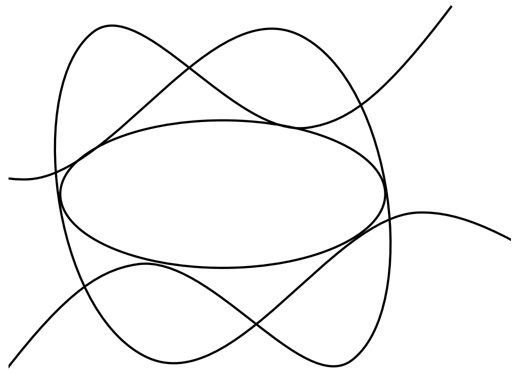
$$19 \neq 20$$

Degenerate a quartic to one with a double point, so its equation is of the form $F = q_2T^2 + q_3T + q_4$, with $q_i \in k[x, y, z]_i$. The projection of the surface to the plane is double, ramified along the sextic

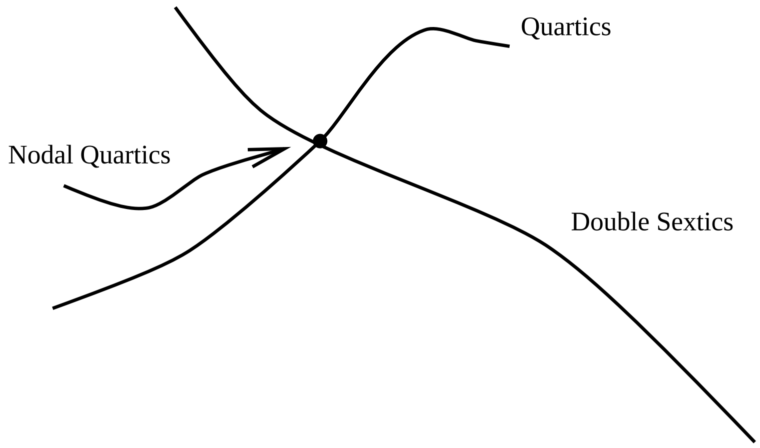
$$4q_2q_4 - q_3^2 = 0$$



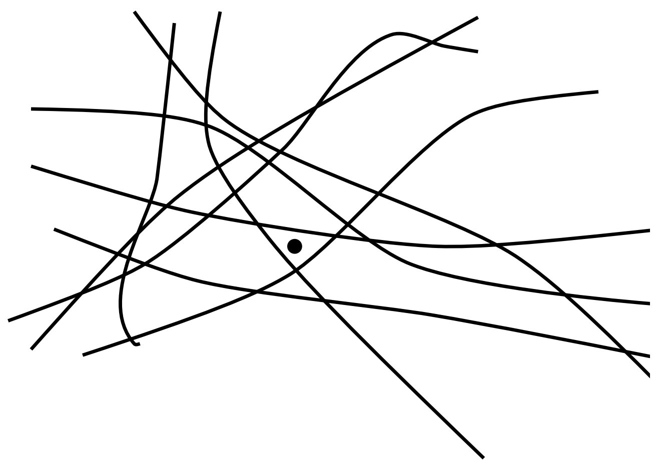
This sextic has the special property of possessing a contact conic: a conic that is tangent to it wherever it meets the sextic.



The whole family of double sextics $w^2 = f_6(X, Y, Z)$ depends on $28 - 3 \times 3 = 19$ parameters. So we found a second family of $K3$-surfaces, again 19-dimensional, intersecting the first family along some 18-dimensional stratum.



This is not the end of the story, rather only the beginning. It tuns out that in all algebraic families one finds 19 parameters. All these algebraic families form an incredibly complicated web inside the 20 dimensional moduli space of $K3$-surfaces.



Can we find some point outside the web? Yes! Take the minimal resolution of $A/(z \mapsto -z)$, where X is a non-algebraic torus.

Let us turn to surfaces of general type. These do not carry a holomorphic vector field, $H^0(\Theta) = 0$. If we can compute $h^2(\Theta)$, we have $h^1(\Theta)$, because we have χ. By duality, $h^2(\Theta) = h^0(\Omega^1(K))$. Usually, this group is non-zero, and we have a problem.

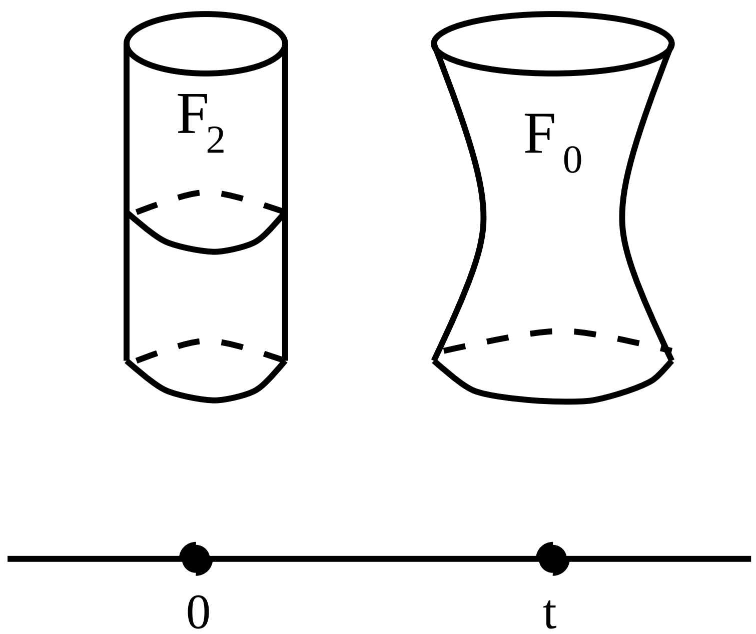
Rational surfaces

For rational surfaces we are in the opposite situation, that $H^0(\Theta)$ is big. For $\mathbb{P}^2$ the dimension is $8 = \dim PGl_3$. The automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ come from Möbius transformations on each factor so $\dim H^0(\Theta) = 3 + 3 = 6$. The surface $\mathbb{F}_2$ is the resolution of the quadric cone. If the vertex of the cone lies in a coordinate point the equation does not contain the corresponding variable. A matrix of an automorphism of $\mathbb{P}^3$ preserving the cone can be obtained from a transformation in the plane preserving the conic and an arbitrary column so $\dim H^0(\Theta) = 3 + 4 = 7$. This is also the dimension for $\mathbb{F}_2$.

To apply our formula for $\chi(\Theta)$ we note that for all surfaces $\mathbb{F}_n$ one has $K^2 = 8$ and $e = 4$ so $\chi(\Theta) = 6$.

	$\chi =$	$h^0(\Theta)$	$-$	$h^1(\Theta)$	$+$	$h^2(\Theta)$
$\mathbb{P}^1 \times \mathbb{P}^1$	$6 =$	6	$-$	0	$+$	0
$\mathbb{F}_2$	$6 =$	7	$-$	?	$+$	?

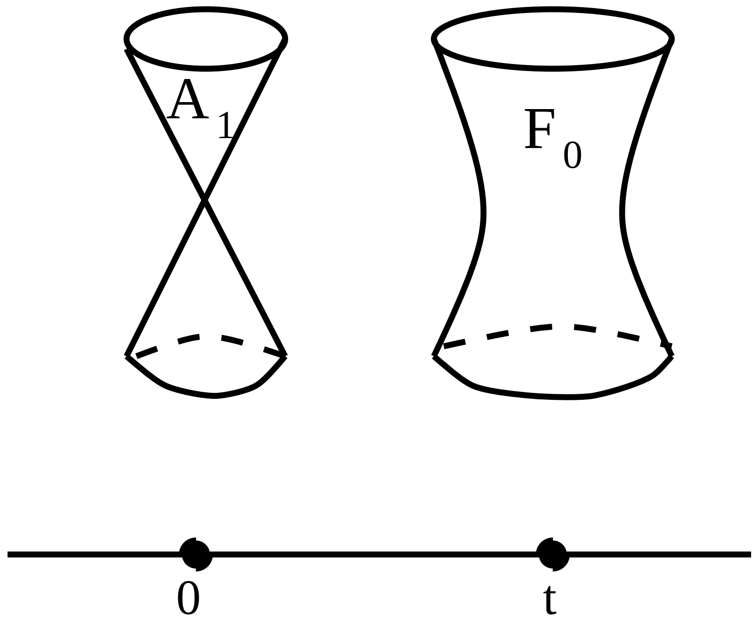
We conclude that $\dim H^1(\Theta_{\mathbb{F}_2}) \geq 1$. In fact it is not difficult to compute all $H^i(\Theta)$ for all surfaces $\mathbb{F}_n$. One has $\dim H^1(\Theta_{\mathbb{F}_2}) = 1$ and therefore $\dim H^2(\Theta_{\mathbb{F}_2}) = 0$. But even without doing this we expect the existence of a 1-parameter deformation with general fibre $\mathbb{P}^1 \times \mathbb{P}^1$.



In fact such a family exists and can be obtained from a small resolution of a deformation of the quadric cone. Consider the family of projective surfaces over $\operatorname{Spec} \mathbb{C}[t]$

$$xy - z^2 + tw^2 = 0 \ .$$

This is the required deformation.


A 1
F 0
0
t

Chapter 15

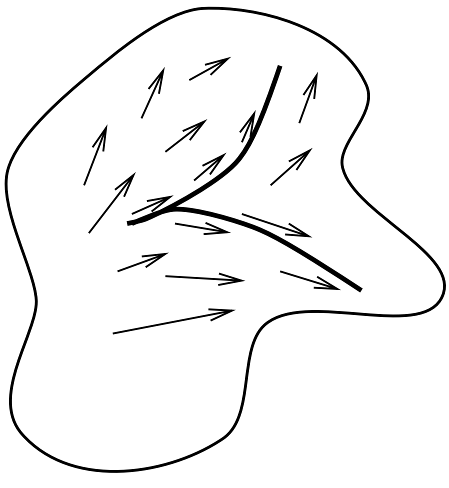
The Cotangent Complex

There is a huge machinery to handle systematically tangent and obstruction spaces of various deformation problems: the calculus of the cotangent complexes.

Consider a singularity X. So we have a ring $R = P/I$, where I is an ideal in the ring $P := k[[\mathbf{X}]]$ of formal power series. in variables $\mathbf{x} = x_1, x_2, \ldots, x_n$. Let $\Omega = \Omega_{P/k} := \oplus_{i=1}^{n} P dx_i$ the module of 1-forms on P and $\Theta := Hom_P(\Omega, P) = \oplus_{i=1}^{n} P\partial/\partial x_i = Der_k(P, P)$ the module of vector fields on P. Recall the exact sequence

$$0 \longrightarrow T^0_{R/k} \longrightarrow \Theta \otimes R \longrightarrow N \longrightarrow T^1_{R/k} \longrightarrow 0$$

Here $T^0_{R/k} = \Theta_{R/k} = Der_k(R, R)$ is the module of vector fields on the singularity X. Geometrically, these are restrictions of vector fields that are tangent to X.



$N = Hom_R(I/I^2, R) = Hom_P(I/R)$ is the normal module of the singularity, and the space of infinitesimal deformations was $T^1_{R/k}$, the cokernel of the natural map $\Theta \otimes R \longrightarrow N$ that maps a θ to the homomorphism $g \in I \mapsto \theta(g) \in R$. For the obstructions there was a space $T^2_{R/k}$.

Well, needless to say there is a T^3 as well! In fact, there is a whole sequence of groups

$$T^0, T^1, T^2, T^3, \ldots, T^k, \ldots$$

These make up some kind of *cohomology theory*, and in fact, these groups are cohomology groups of a certain complex, the *cotangent complex* $\mathbb{L}$.

How do we construct such a cohomology theory? Recall the construction of the derived functors $\mathrm{Ext}^*(M,N)$ of $\mathrm{Hom}(M,N)$. It consists of taking the following three steps.

(1) take a free resolution of M as an R-module.

$$\ldots \longrightarrow F^2 \longrightarrow F^1 \longrightarrow F^0 \longrightarrow M \longrightarrow 0$$

(2) apply the functor $\mathrm{Hom}(-,N)$ to the resolution F^*. We get a complex

$$\mathrm{Hom}(F^0,N) \longrightarrow \mathrm{Hom}(F^1,N) \longrightarrow \mathrm{Hom}(F^2,N) \longrightarrow$$

(3) take the homology of this complex

$$\overset{k}{\mathrm{Ext}}(M,N) = H^k(\mathrm{Hom}(F^*,N))$$

What we want to do now is the "resolve" the ring R and replace it by some smooth algebra, that is *homologically* the same as R. Then we apply the functor 'taking one-forms' or 'taking vector fields'. Finally, we take homology groups and obtain the tangent and cotangent homology.

Definition 15.1. A *graded-commutative* k-algebra A is a direct sum of k-modules

$$A = \oplus_{i\in\mathbb{Z}} A^i$$

with a *graded commutative* product:

$$ab = (-1)^{|a||b|}ba$$

Here $|a|$ denotes the degree of the element a, that is, $a \in A^{|a|}$.

A *differential graded algebra*, DG-algebra for short, is a pair consisting of a graded commutative algebra A, together with a *differential*

$$\partial : A \longrightarrow A.$$

This differential is required to have the properties
1) $\partial \circ \partial = 0$.
2) $\partial : A^p \longrightarrow A^{p+1}$.
3) $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$
A map with these last two properties is called a *derivation of degree 1.*

We will only consider A's with $A^k = 0$ for $k > 0$. We then can consider (A,∂) as a complex of the form

$$\longrightarrow A^{-2} \xrightarrow{\partial} A^{-1} \xrightarrow{\partial} A^0 \longrightarrow 0.$$

Remarks: • If we are given symbols z_i, i in some index set I, with degrees $|z_i| \in \mathbb{Z}$, then one can consider the *free* graded commutative algebra on the generators z_i

$$A = k[\mathbf{z}] := k[z_i\ i \in I].$$

• Any commutatitive k-algebra R can be considered as a DG-algebra by putting R in degree zero:

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow R \longrightarrow 0$$

For any DG-algebra A, the cohomology-object

$$H(A,\partial) = \ker(\partial)/\mathrm{Im}(\partial)$$

is a graded commutative algebra in a natural way:

$$H^p(A,\partial) = ker(A^p \longrightarrow A^{p+1})/Im(A^{p-1} \longrightarrow A^p)$$

Definition 2:
A *resolvent* for a k-algebra R is a map
$$A \longrightarrow R$$
where (A, ∂) is a *free* DG-algebra, and the map induces an isomorphism
$$H(A, \partial) = R$$
Put in another way, we require that the complex A resolves R, that is, the sequence
$$\longrightarrow A^{-2} \xrightarrow{\partial} A^{-1} \xrightarrow{\partial} A^0 \longrightarrow R.$$
is exact.

Example 15.2. Let $f \in P := k[\mathbf{x}] = k[x_1, x_2, \dots, x_n]$ and $R = P/(f)$ a hypersurface ring. We consider
$$A = k[\mathbf{x}, e]$$
where e is an *extra generator* of degree -1. So we have $e^2 = 0$.
We define the differential ∂ as follows:
$$\partial x_i = 0, \qquad \partial e = f$$
Written as a complex, this is:
$$\begin{array}{ccc} -1 & & 0 \\ \hline Pe & \xrightarrow{\partial} & P \\ a.e & \longmapsto & \partial(a)e + a\partial(e) = a.f \end{array}$$

Claim: (A, δ) is a resolvent for R.

Let us try to do this with more equations.
$$I = (f_1, f_2, \dots, f_p) \subset P = k[\mathbf{x}]$$
and put $R := P/I$. Consider the free graded commutative algebra
$$A := k[\mathbf{x}, e_1, e_2, \dots, e_p] = k[\mathbf{x}, \mathbf{e}],$$
where we put the e_i's in degree -1, so
$$e_i e_j = -e_j e_i.$$
We define the differential by putting
$$\partial x_i = 0, \;\; i = 1, 2, \dots, n \quad , \quad \partial e_i = f_i, \;\; i = 1, 2, \dots, p.$$
As a complex, (A, ∂) is just isomorphic to the *Koszul-complex* on the elements f_i:
$$A^{-k} = \bigoplus_{i_1 < i_2 < \dots < i_k} P\, e_{i_1} e_{i_2} \dots e_{i_k} \approx P^{\binom{p}{k}}\ .$$

Clearly, $H^0(A, \delta) = R$. So let us look at $H^{-1}(A, \partial)$.

$$\begin{aligned} Ker(\bigoplus Pe_i \longrightarrow P) &= \{\sum_i r_i e_i \mid \partial(\sum_i r_i e_i) = 0\} \\ &= \{\sum_i r_i e_i \mid \sum_i r_i f_i = 0\} \\ &=: \mathscr{R} \end{aligned}$$

So $\mathscr{R}$ is precisely the the module of *relations* between the chosen generators f_i. What is $Im(A^{-2} \xrightarrow{\partial} \oplus_i A^{-1})$? Well, $\partial(e_i e_j) = \partial(e_i)e_j - e_i\partial(e_j) = f_i e_j - f_j e_i$, which is called the (i,j)'th Koszul relation. So the image can be identified with the module $\mathscr{R}_0$ of *Koszul relations.* And thus

$$H^{-1}(A,\partial) \approx \mathscr{R}/\mathscr{R}_0$$

It is well-known that the Koszul complex is exact precisely when the f_i form a regular sequence, that is, when R is a complete intersection ring. So in that case the Koszul complex is a resolvent for R. But in general, $\mathscr{R}/\mathscr{R}_0$ will be non-zero, and the above complex is not a resolvent for our ring R. Even worse, there will be H^{-3}, etc.

What can we do to get rid of this unwanted cohomology groups? Choose generators for the $P = A_0$-module $H^{-1}(A,\partial)$ represented by relations

$$\sum_j \rho_{ij} e_j, \quad i = 1, 2, \ldots, r$$

We enlarge our DG-algebra $P[\mathbf{e}]$ by putting in extra generators ρ_i with $|\rho_i| = -2$. That is, we consider $A = P[\mathbf{e},\rho]$ and define

$$\partial(\rho_i) = \sum_j \rho_{ij} e_j$$

To define ∂ on other new elements of A, like $\rho_i.e_j$ we use Leibniz rule: $\partial(\rho_i e_j) := \partial(\rho_i)e_i + \rho_i\partial(e_j)$, etc. The complex now looks like:

$$\begin{array}{lclcl}
 & -2 & & -1 & 0 \\
\hline
\cdots \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow \; A^0 \\
\cdots \longrightarrow & \oplus_{i<j} P e_i e_j \bigoplus \oplus_i P\rho_i & \xrightarrow{\partial} & \oplus_i P e_i & \longrightarrow P
\end{array}$$

For this new DG-algebra $A = P[\mathbf{e},\rho]$ it holds *by construction* $H^0(A,\partial) = R$, $H^{-1}(A,\partial) = 0$.

Of course, now there will in general be non-zero $H^{-2}(A,\partial)$. But the above process can be repeated. We choose generators $\tau_1, \tau_2, \ldots, \tau_s$ for $H^{-2}(A,\partial)$ and extend the algebra to $P[\mathbf{e},\rho,\tau]$. We can go on with this forever, and create some huge DG-algebra

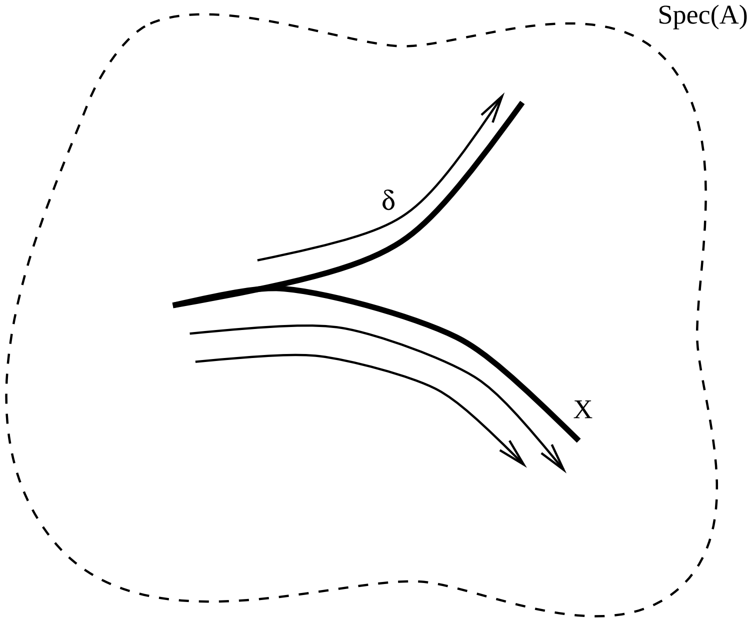
$$A = P[\mathbf{e},\rho,\tau,\ldots]$$

which has the property that $H^0(A,\partial) = R$, $H^k(A,\partial) = 0$ for $k \neq 0$.

Remarks:

- The construction depends on many choices, but one senses that the final object (A,∂) is essentially unique.
- Rather than working with polynomial rings $k[\mathbf{x}]$ one can start with power series rings $k[[\mathbf{x}]]$ or $k\{\mathbf{x}\}$ and arrive at a resolvent of the form $A = k[[\mathbf{x}]][\mathbf{e},\rho,\tau,\ldots]$.
- As a free object, the ring A does not contain much information as such. All interesting information about our original ring gets packed into the differential ∂.
- Geometrically, the resolvent is an infinite dimensional superspace, together with an odd vector

field ∂ on it, whose homology ('fixed point set') is our original space.



Let us label all our variables $z_i = \mathbf{x}, \mathbf{e}, \rho, \tau, \ldots$ by some index set I, so that $A = k[z_i \; i \in I]$. Consider the module Ω_A of differentials on A. It is just

$$\Omega_A := \bigoplus_{i \in I} A dz_i$$

We give degrees by putting $|dz_i| := |z_i$. We have the universal derivation defined by

$$d : A \longrightarrow \Omega_A \;\; ; \;\; z_i \mapsto dz_i$$

and extended using Leibniz rule $d(ab) = (da)b + (-1)^{|a|} adb$ The differential ∂ on A descends to a differential on Ω_A:

$$\partial : \Omega_A \longrightarrow \Omega_A \;\; ; \;\; \partial(dz_i) := d(\partial z_i)$$

In this way we obtain a complex

$$(\Omega_A, \partial).$$

Similarly, we have the complex

$$\Theta_A = Hom_A(\Omega_A, A) = Der_A(A, A) = \bigoplus_{i \in I} A \frac{\partial}{\partial z_i}$$

$(|\frac{\partial}{\partial z_i}| = -|z_i|.)$ Elements in degree k are maps $\delta : A \longrightarrow A$ such that $\delta : A^i \longrightarrow A^{i+k} \quad \delta(ab) = \delta(a)b + (-1)^{|a|k} a\delta(b)$ and are called a derivations of degree k. The operation of graded commutator

$$[\alpha, \beta] := \alpha \circ \beta - (-1)^{|a||b|} \beta \circ \alpha$$

gives Θ the structure of a super Lie-algebra. That is, one has:

$$[\alpha, \beta] = -(-1)^{|a||b|} [\beta, \alpha]$$

and the graded Jacobi-identity holds:

$$(-1)^{|\alpha||\gamma|} [\alpha, [\beta, \gamma]] + (-1)^{|\beta||\alpha|} [\beta, [\gamma, \alpha]] + (-1)^{|\gamma||\beta|} [\gamma, [\alpha, \beta]] = 0$$

The differential δ is a particular derivation of degree 1. Commuting with it defines a map

$$\nabla : Der(A, A) \longrightarrow Der(A, A) \;\; ; \;\; \alpha \mapsto [\delta, \alpha]$$

It follows from the graded Jacobi-identity that

$$\nabla \circ \nabla = 0$$

Hence, $(Der(A, A), \nabla)$ is a complex, in fact the dual to (Ω_A, ∂).

Definition:

$$H^{-i}(\Omega_A, \partial) =: T_i^{R/k}$$
$$H^i(\Theta_A, \nabla) =: T^i_{R/k}$$

The bracket $[-,-]$ on $Der(A, A) = \Theta_A$ induces a bracket

$$T^p \times T^q \longrightarrow T^{p+q}$$

giving

$$T^* := \sum_{i=0}^{\infty} T^i$$

the structure of a graded super Lie-algebra.

Definition: The cotangent complex is the complex of free R-modules

$$\mathbb{L}_{R/k} := \Omega \otimes_A R = \bigoplus_{i \in I} R dz_i$$

If M is any R-module one defines

$$T_i(R/k, M) := H^{-i}(\mathbb{L}_{R/k} \otimes M)$$

$$T^i(R/k, M) := H^i(Hom_R(\mathbb{L}_{R/k}, M))$$

These are called the i-th tangent homology and cohomology of R (over k with values in M.

Example: We look again at a hypersurface: $R = P/(f)$, $P = k[\mathbf{x}]$ $A = P[e]$, $\partial(e) = f$. Put $\Omega := \bigoplus_{i=1}^n P dx_i$. The complex Ω_A : looks like

$$\begin{array}{ccccc} -2 & & -1 & & 0 \\ \hline Pede & \longrightarrow & Pde \oplus e\Omega & \longrightarrow & \Omega \end{array}$$

The differential works as follows:

$$\partial(de) = d(\partial(e)) = df = \sum_i \frac{\partial f}{\partial x_i}$$

$$\partial(edx) = \partial(e)dx - e\partial(dx) = f dx$$

We see: $T_0 = \Omega_{R/k}$, $T_i = 0$ for $i \neq 0$. With $\Theta := \bigoplus_{i=1}^n P \frac{\partial}{\partial x_i}$, the complex Θ_A : looks like:

$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ \hline Pe\frac{\partial}{\partial e} & \longrightarrow & Pe\frac{\partial}{\partial e} \oplus \Theta & \longrightarrow & P\frac{\partial}{\partial e} \end{array}$$

As homology we find: $T^0 = \Theta_{R/k}$, $T^1 = P/(f, \partial f/\partial x_i)$, as it should be.

The cotangent complex is obtained by tensoring Ω_A with R. The effect is putting all new elements, like e equal to zero an d computing module f in P. So we have:

$\mathbb{L}_{R/k}$

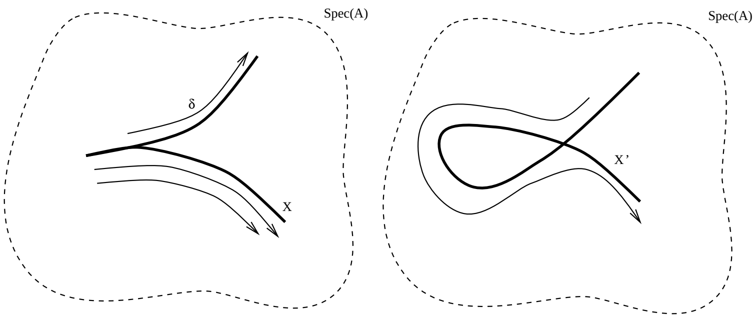
$$\begin{array}{ccccc} -2 & & -1 & & 0 \\ \hline 0 & \longrightarrow & Rde & \longrightarrow & \Omega \otimes R \end{array}$$

The dual complex is $Hom_R(\mathbb{L}_{R/k,R})$:

$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ \hline 0 & \longrightarrow & \Theta \otimes R & \longrightarrow & R\frac{\partial}{\partial e} \end{array}$$

The Deformation Equation

Starting from a k-algebra R we constructed a resolvent A, with a super vector field $\partial : A \longrightarrow A$. The cohomology of this field was R.



As R and ∂ are sort of equivalent data, it is natural to expect a close relation between deformations of $X = Spec(R)$ and deformations of the differential ∂: a deformed differential ∂' defines $X' = spec(R')$, where $R' = ker(\partial')/Im(\partial')$.

$\partial' := \partial + \omega$ is a differential if

$$\begin{aligned} 0 &= \partial' \circ \partial' \\ &= (\partial + \omega) \circ (\partial + \omega) \\ &= \omega \circ \partial + \partial \circ \omega + \omega \circ \omega \\ &= [\omega, \partial] + \tfrac{1}{2}[\omega, \omega] \\ &= \nabla\omega + \tfrac{1}{2}[\omega, \omega] \end{aligned}$$

This last equation $\boxed{\nabla\omega + \frac{1}{2}[\omega,\omega] = 0}$ occurs over and over in mathematics. We call it the *deformation equation*. We will see that it defines the semi-universal deformation of X. We remark that the first order term is $\nabla\omega = 0$, which means that $\omega \in H^1(Der(A, A)) = T^1_{R/k}$.

Excercise: If $\omega \in T^1_{R/k}$, show that $[\omega, \omega] \in T^2_{R/k}$

Chapter 16

Lichtenbaum-Schlessinger Complex

For applications in deformation theory one usually only needs T^0,T^1 and T^2. To get these groups, one does not need to construct a complete resolvent; it suffices to work with with the truncated complex. Let $\mathbb{L}_{R/k}$ be the cotangent complex. Let us look at the beginning:

$$\begin{array}{ccccccc} & -2 & & -1 & & 0 \\ \hline \longrightarrow & (\mathbb{L}_{R/k})_{-1} & \longrightarrow & (\mathbb{L}_{R/k})_{-1} & \longrightarrow & (\mathbb{L}_{R/k})_0 \\ & \oplus_{i=1}^r R d\rho_i & \longrightarrow & \oplus R de_i & \longrightarrow & \Omega \otimes R \end{array}$$

Under ∂ the element $d\rho_i$ is mapped to a system of generators $\rho_i j e_j$ for the module $\mathscr{R}/\mathscr{R}_0$ of relations mod Koszul relations between the f_i. When we mod out $\mathrm{Im}(\partial : \mathbb{L}_{-3} \longrightarrow \mathbb{L}_{-2})$ we get what is called the *Lichtenbaum-Schlessinger* complex.

$$\begin{array}{ccccc} -2 & & -1 & & 0 \\ \hline \mathscr{R}/\mathscr{R}_0 & \longrightarrow & \oplus R\, de_i & \longrightarrow & \Omega \otimes R \\ r & \longmapsto & \sum r_i\, de_i & & \\ & & de_i & \longmapsto & df_i \end{array}$$

The groups T^0,T^1 and T^2 can be calculated by taking the dual of this complex. It reads

$$\begin{array}{ccccc} -2 & & -1 & & 0 \\ \hline \Theta \otimes R & \longrightarrow & R\frac{\partial}{\partial e_j} & \longrightarrow & \mathrm{Hom}_R(\mathscr{R}/\mathscr{R}_0, R) \\ \theta & \longmapsto & \sum_i \theta(f_i)\frac{\partial}{\partial e_i} & & \\ & & \frac{\partial}{\partial e_j} & \longmapsto & (\rho_i \mapsto \sum_j \rho_{ij}) \end{array}$$

We see that the homology group in degree 0 is $T^0 = \Theta_R/k = \{\theta \in \Theta \otimes R \mid \theta(f_i) \subset (f_1, \ldots, f_k)\}$. Elements $\sum_i g_i \frac{\partial}{\partial e_i}$ that map to zero in $Hom_R(\mathscr{R}/\mathscr{R}_0, R)$ correspond precisely elements of $f_i \mapsto g_i$ of $N = Hom_R(I/I^2, R)$. We see that T^0, T^1 and T^2 are indeed the same as we defined before by more ad hoc definitions.

The SOUP

We defined the cotangent complex for algebras over k, but of course one can work over any base ring S. The cotangent complex $\mathbb{L}_{R/S}$ can be defined in great generality, and so we obtain modules $T^i(R/S, M)$ There is a list of useful properties, which can be taken as axioms, which everyone should know.
Here comes this Set Of Useful Properties:

(1) *cohomology theory*
Any short exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

gives rise to a long exact (co)-homology sequence of R-modules involving the T^i or T_i's:

$$0 \longrightarrow T^0(R/S, M') \longrightarrow T^0(R/S, M) \\ \longrightarrow T^0(R/S, M'') \longrightarrow T^1(R/S, M') \longrightarrow T^1(R/S, M) \longrightarrow \dots$$

(2) *spectral sequence*
There is a spectral sequence relating T^i and T_i.

$$E^2_{p,q} := Ext^p_R(T_q(R/S, R), M) \Longrightarrow T^{p+q}(R/S, M).$$

(3) $i = 0$

$$\begin{aligned} T^0(R/S, M) &= \Omega_{R/S} \otimes M \\ T_0(R/S, M) &= \mathrm{Hom}(\Omega_{R/S}, M) \end{aligned}$$

(4) *Vanishing*
If R is a *smooth* S-algebra, then

$$T_i(R/S, M) = T^i(R/S, M) = 0 \qquad i \geq 1$$

So the T's are concentrated at the singularities.

(5) $i = 0$

$$T_0(R/S, M) = \Omega_{R/S} \otimes M, \qquad T^0(R/S, M) = \mathrm{Hom}(\Omega_{R/S}, M) = \operatorname*{Der}_S(R, M)$$

(6) *(co)-normal*
If $P \longrightarrow R$ is surjective map of S-algebras, with kernel I, then $T_0(P/R, M) = T^0(P/R, M) = 0$. Furthermore,

$$\begin{aligned} T_1(P/R, M) &= I/I^2 \otimes_P M \\ T^1(P/R, M) &= Hom_P(I/I^2, M) \end{aligned}$$

(7) *Base change*
If R is a flat S-module, and R' obtained by base-changing from a map $S \longrightarrow S'$, (i.e. $R' = R \otimes_S S'$), then

$$\mathbf{L}_{R'/S'} = \mathbf{L}_{R/S} \otimes_R R'$$

From this one gets isomorphisms

$$T^i(R'/S', M') = T^i(R/S, M').$$

If moreover in this situation $S \longrightarrow S$" is flat, you can pull out:

$$T^i(R'/S', M \otimes_R R') = T^i(R/S, M) \otimes_R R'\ .$$

(8) *Zariski-Jacobi sequence*
For any map of S-algebras $P \longrightarrow R$ there is an exact sequence of complexes

$$0 \longrightarrow \mathbb{L}_{P/S} \otimes R \longrightarrow \mathbb{L}_{R/S} \longrightarrow \mathbb{L}_{R/P} \longrightarrow 0.$$

Associated to this sequence and an P-module M, there are interesting long exact sequences:

$$\dots \longrightarrow T_{i+1}(R/P, M) \longrightarrow T_i(P/S, M) \longrightarrow T_i(R/S, M) \longrightarrow T_i(R/P, M) \longrightarrow \dots$$

$$\dots \longrightarrow T^i(R/P, M) \longrightarrow T^i(R/S, M) \longrightarrow T^i(P/S, M) \longrightarrow T^{i+1}(R/P, M) \longrightarrow \dots$$

The Zariski-Jacobi sequences sort of characterise the T_i and T^i's. It is the derived version of the chain rule.

Example Let P be smooth over $S = k$, $R = P/I$ and $M = R$. The Zariski-Jacobi sequence reads:

$$\begin{aligned} 0 \longrightarrow T^0(R/P,R) \longrightarrow T^0(R/k,R) \longrightarrow T^0(P/k,R) \\ \longrightarrow T^1(R/P,R) \longrightarrow T^1(R/k,R) \longrightarrow T^1(P/k,R) \longrightarrow T^2 \cdots \end{aligned}$$

The first module is zero, because $P \twoheadrightarrow R$. The second module is just $\Theta_{R/k}$, the vector fields on X. The third module is $\Theta_{P/k} \otimes R$, the fourth term $T^1(R/P,R) = Hom(I/I^2,R) = N$, the fifth term is just $T^1(R/k)$. The sixth term is $T^1(P/k,R) = 0$, because P is smooth over k. Hence, the sequence reduces to the usual sequence defining T^1. Moreover, we get isomorphisms for $i \geq 2$.

$$T^i(R/P,R) \xrightarrow{\sim} T^i(R)$$

Chapter 17

Spectral Sequences

We have defined the cotangent complex for rings. Let now X be any scheme or analytic space and $p \in X$ a point. Then $\mathscr{O}_{X,p}$ is a ring and $\mathbb{L}_{X,p}$ is a $\mathscr{O}_{X,p}$-module. Considering them altogether gives us a complex of $\mathscr{O}_X$-sheaves. In fact one globalises as usual; in the algebraic case one rather considers the affine open sets U and their rings $\mathscr{O}(U)$.

Once we have the complex of sheaves $\mathbb{L}^{\cdot}_X$ we define the T^i by taking cohomology:

$$\mathbb{T}^i := \mathbb{H}^i(X, \mathbb{L}^{\cdot}_X) \,.$$

The symbol 'ℍ' stands for what is called hypercohomology.

One can define hypercohomolgy with a Čech covering. Cover X with open sets U_i. For a sheaf $\mathscr{F}$ on X one has the Čech cochains $C^p(\mathscr{F}) = \prod \Gamma(U_{i_0}, \dots, U_{i_p}, \mathscr{F})$ with differential δ. We define a double complex

$$K^{p,q} = C^p(\mathbb{L}^q)$$

with differentials d coming from $\mathbb{L}^{\cdot}_X$ and δ.

$$\begin{array}{|cccc}
\vdots & \vdots & \vdots & \iddots \\
C^0(\mathbb{L}^2) & C^1(\mathbb{L}^2) & C^2(\mathbb{L}^2) & \dots \\
C^0(\mathbb{L}^1) & C^1(\mathbb{L}^1) & C^2(\mathbb{L}^1) & \dots \\
C^0(\mathbb{L}^0) & C^1(\mathbb{L}^0) & C^2(\mathbb{L}^0) & \dots \\
\hline
\end{array}$$

The hypercohomology $\mathbb{H}^i(\mathbb{L}^{\cdot}_X)$ is now by definition the cohomology of the associated single complex

$$K^n = \bigoplus_{p+q=n} K^{p,q}$$

which has differential $D = d \pm \delta$: to make it into a complex one defines the differential in the p-direction by $(-1)^q\delta\colon K^{pq} \to K^{p+1,q}$.

Let us first look at $H^0(\mathbb{L})$:

$$\begin{tikzcd}
C^0(\mathbb{L}^1) & \\
C^0(\mathbb{L}^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathbb{L}^0)
\end{tikzcd}$$

For a cochain $k \in K^0 = C^0(\mathbb{L}^0)$ the condition $D\,k = 0$ is equivalent with the two conditions $d\,k = 0$ and $\delta\,k = 0$. The last one says that k is a global section of $\mathbb{L}^0$ while $d\,l = 0$ expresses the fact that $k_{|U_i}$ takes values in $T^0_{U_i}$, because on U_i the complex $\mathbb{L}^0$ has zero'th cohomology $T^0_{U_i}$. In fact one obtains in this way a sheaf which we denote by $\mathscr{T}^0$ or $\mathscr{T}^0_X$. The higher cohomology of the complex leads in the same way to cohomology sheaves $\mathscr{T}^i$. So our element k with $D\,k = 0$ is an element of $H^0(X, \mathscr{T}^0)$ and we have shown that

$$\mathbb{T}^0_X = H^0(X, \mathscr{T}^0)\ .$$

To find $\mathbb{T}^1$ we look at $C^0(\mathbb{L}^1)$ and $C^1(\mathbb{L}^0)$.

hier eventueel een diagramjaagd

Claim 17.1. *There is an exact sequence*

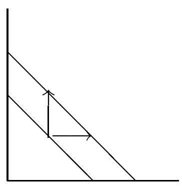
$$0 \longrightarrow H^1(\mathscr{T}^0) \longrightarrow \mathbb{T}^1 \longrightarrow H^0(\mathscr{T}^1) \longrightarrow H^2(\mathscr{T}^0)$$

There exists an useful tool to organise such computations: a *spectral sequence*. In most of the applications lots of things become zero which makes it is easy to compute with spectral sequences. But the general formalism covers all cases and the maps involved can be quite hard to compute.

We start by defining E^{pq}_0 as something which is isomorphic to K^{pq} but defined differently:

$$E^{pq}_0 = \frac{K^{pq} + K^{p+1,q-1} + \cdots}{K^{p+1,q-1} + \cdots} \cong K^{pq}$$

The map $D\colon K^{pq} \to K^{p,q+1} \oplus K^{p+1,q}$ induces a map $d_0\colon E^{pq}_0 \to E^{p+1,q}_0$.



Now we can compute cohomology and define E^{pq}_1 as the pth cohomology of the complex $(E^{p,\cdot}_0, d_0)$. The differential D still induces a differential d_1, this time as a map $E^{pq}_1 \to E^{p,q+1}_1$. And this process can be repeated: $E^{\cdot\cdot}_{r+1}$ is the cohomology of $(E^{\cdot\cdot}_r, d_r)$ with $d_r\colon E^{pq}_r \to E^{p+r,q-r+1}_r$.

At this point a number of things have to be checked, e.g., that D really induces the said differentials d_r. And it is best to do this yourself. The answer to these exercises can be found in any good book on homological algebra. You should be warned that there exists also a slick abstract approach to spectral sequences using 'exact couples'.

We obtain in our case that $E^{pq}_0 \cong C^p(\mathbb{L}^q)$, $E^{pq}_1 = C^p(\mathscr{T}^q)$ and $E^{pq}_2 = H^p(X, \mathscr{T}^q)$. Often one sees a spectral sequence written in the form that only the E^{pq}_2-term is given. Note that now Čech cohomology is no longer mentionned: we can compute the sheaf cohomology $H^p(X, \mathscr{T}^q)$ by any means we like.

Example 17.2. Consider the case that X has only complete intersection singularities. Then $\mathscr{T}^i_X = 0$ for $i > 1$ and the E_2-term of our spectral sequence consists of two non-trivial rows.

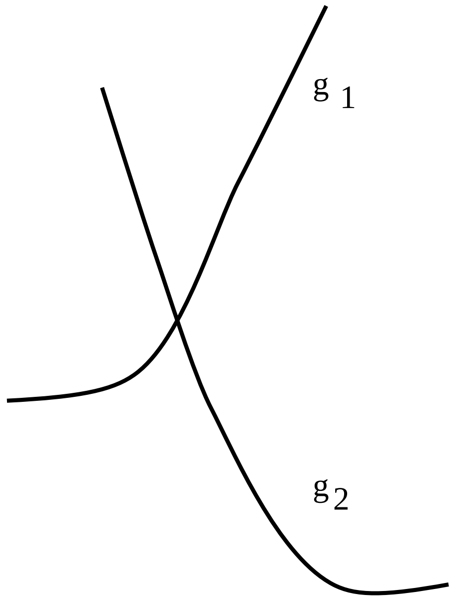
$$\begin{array}{|cccc}
\vdots & \vdots & \vdots & \iddots \\
0 & 0 & 0 & \dots \\
H^0(\mathscr{T}^1) & H^1(\mathscr{T}^1) & H^2(\mathscr{T}^1) & \dots \\
H^0(\mathscr{T}^0) & H^1(\mathscr{T}^0) & H^2(\mathscr{T}^0) & \dots \\
\hline
\end{array}$$

The maps d_2 are now maps $d_2\colon H^i(sT^1) \to H^{i+2}(\mathscr{T}^0)$, and in (E_3, d_3) the maps d_3 go two steps down, so are necessarily zero maps. Therefore $E_3 = E_4 = \ldots = E_\infty$ and one says that the spectral sequence converges, to $\mathbb{T}_X$.

We conclude that $\mathbb{T}^0 = H^0(\mathscr{T}^0)$ and the remaining $\mathbb{T}^i$ occur in a long exact sequence:

$$0 \longrightarrow H^1(\mathscr{T}^0) \longrightarrow \mathbb{T}^1 \longrightarrow H^0(\mathscr{T}^1) \longrightarrow H^2(\mathscr{T}^0) \longrightarrow \mathbb{T}^2 \longrightarrow \cdots$$

Example 17.3. Consider the case of a curve with isolated singular points.



The sheaves $\mathscr{T}_X^i$, $i \geq 1$ are concentrated at the singular points, so $H^j(\mathscr{T}_X^i) = 0$, $i, j \geq 1$. We get a short exact sequence

$$0 \longrightarrow H^1(\mathscr{T}^0) \longrightarrow \mathbb{T}_X^1 \longrightarrow H^0(\mathscr{T}^1) \longrightarrow 0$$

and isomorphisms

$$\mathbb{T}_X^i = H^0(\mathscr{T}_X^i), \qquad i \geq 2$$

In particular, if all the singularities of the curve are complete intersections, then $H^0(\mathscr{T}_X^2) = 0$, hence $\mathbb{T}^2 = 0$: there are no obstructions, so $\mathrm{Def}(X)$ is smooth. Moreover, all local deformations of the singularities can be globalised to deformations of X.

Chapter 18

Cotangent Complex II

Relative case

Associated to a map $f : X \longrightarrow Y$ of complex spaces, there are (at least) six a priori different deformation problems one can think of. These are

(1) $\mathrm{Def}(X \longrightarrow Y)$

(2) $\mathrm{Def}(X/Y)$

(3) $\mathrm{Def}(X\backslash Y)$

(4) $\mathrm{Def}(X)$

(5) $\mathrm{Def}(Y)$

(6) $\mathrm{Def}(f)$

In the first case $\mathrm{Def}(X \longrightarrow Y)$ we deform everything: X, Y and the map between them. So objects of $\mathrm{Def}(X \longrightarrow Y)(S)$ are (isomorphism classes, of course) of f maps $X_S \overset{f_S}{Y_S}$ where X_S and Y_S are S-flat and restricting to $X \xrightarrow{f} Y$ over the special point. Similarly, $\mathrm{Def}(X/Y)$ consists of deformations *over* Y, that is Y is deformed trivially. $\mathrm{Def}(X/Y)(S)$ consists of maps $X_S \xrightarrow{f_S} Y \times S$, where X_S is S-flat, etc. We leave it to the reader to think of the meaning of the other cases. In each of these cases there exists a cotangent complex associated to the deformation problem. For $\mathrm{Def}(X \longrightarrow Y)$ we have $\mathbb{L}_{X \longrightarrow Y}$, for $\mathrm{Def}(X/Y)$ there is $\mathbb{L}_{X/Y}$.

Let us take a closer look at $\mathbb{L}_{X/Y}$. Is is a complex of sheaves on X. Roughly speaking, it is constructed as in the local case: at $x \in X$ it is the cotangent complex

$$\mathbb{L}_{R/S}$$

where $R = \mathcal{O}_{(X,x)}$ and $S = \mathcal{O}_{(Y,y)}$ In particular, the cohomology sheaves $\mathscr{T}^i_{X/Y}$ of $\mathbb{L}_{X/Y}$ have stalks

$$(\mathscr{T}^i_{X/Y})_p = T^i(\mathscr{O}_{(X,p)}/\mathscr{O}_{Y,f(p)}, \mathscr{O}_{(X,p)})$$

There are also global T^i's

$$\mathbb{T}^i_{X/Y} := \mathbb{H}^i(\mathbb{L}_{X/Y})$$

These hypercohomology groups can be computed in the usual way by a local to global spectral sequence. These groups have for $k = 0, 1, 2$ obvious interpretations as first order automorphisms, deformations and obstructions of X *over* Y. That is, we deform $X \longrightarrow Y$, **but** Y *is kept fixed.*

Let us look at some important special cases.

Case 1. $Y = S$ is smooth 1-dimensional, and $f : X \longrightarrow S$ is flat. Let t be the local parameter for $\mathscr{O}_(S, 0) = \mathbb{C}\{t\}$ and $X_0 = f^{-1}(0)$ the zero-fibre. The exact sequence

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{t} \mathscr{O}_X \longrightarrow \mathscr{O}_{X_0} \longrightarrow 0$$

We get a long exact sequence

$$\ldots \longrightarrow \mathbb{T}^i_{X/S}(\mathscr{O}_X) \xrightarrow{t} \mathbb{T}^i_{X/S}(\mathscr{O}_X) \longrightarrow \mathbb{T}^i(\mathscr{O}_{X_0}) \longrightarrow \ldots$$

By base change, $\mathbb{T}^i_{X/S}(\mathscr{O}_{X_0}) = T^i_{X_0}$, so we get a sequence relating deformations of a fibre to deformations of the family.

Case 1. When $f : X \longrightarrow Y$ is an embedding.

local situation

$\mathscr{T}^i_{X/Y} = 0$, $\mathscr{T}^1_{X/Y} \approx \mathscr{N}_{X/Y}$ is the normal sheaf of $X \hookrightarrow Y$. The Zariski-Jacobi sequence of complexes

$$\mathbb{L}_Y \otimes \mathcal{O}_X \longrightarrow \mathbb{L}_X \longrightarrow \mathbb{L}_{X/Y}$$

gives the usual Zariski-Jacobi sequence of $\mathscr{T}^i$'s:

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_Y \otimes \mathscr{O}_X \longrightarrow \mathscr{N}_{X/Y} \longrightarrow \mathscr{T}^1_X \longrightarrow \mathscr{T}^1_Y(\mathscr{O}_X) \longrightarrow \ldots$$

$$\ldots \longrightarrow \mathscr{T}^{k-1}(\mathscr{O}_X) \longrightarrow \mathscr{T}^k_{X/Y} \longrightarrow \mathscr{T}^k_X \longrightarrow \mathscr{T}^k_Y(\mathscr{O}_X) \longrightarrow \ldots$$

We see that if Y is *smooth*, we get isomorphisms

$$\mathscr{T}^k_{X/Y} = \mathscr{T}^k_X \;\; k \geq 2$$

global situation

The global Zariski-Jacobi sequence :

$$0 \longrightarrow \mathbb{T}^0_X \longrightarrow \mathbb{T}^0_Y(\mathscr{O}_X) \longrightarrow \mathbb{T}^1_{X/Y} \longrightarrow \mathbb{T}^1_X \longrightarrow \mathbb{T}^1_Y(\mathscr{O}_X) \longrightarrow \mathbb{T}^2_{X/Y} \longrightarrow \ldots$$

might look a little bit unfamiliar at first. The most interesting part of the sequence seems to be the map

$$\mathbb{T}^1_{X/Y} \longrightarrow \mathbb{T}^1_X$$

It maps a deformation of X *in* Y to the deformation of just X. This implies: if $\mathbb{T}^1_Y(\mathscr{O}_X) = 0$, then all deformations of X can be realised inside Y.

local-to-global

Let us see how we can compute the $\mathbb{T}^k_{X/Y}$. Of course, there is again a spectral sequence doing the job:

$$E_2^{pq} := H^p(\mathscr{T}^q_{X/Y}) \Longrightarrow \mathbb{T}^{p+q}_{X/Y}$$

The diagram looks like:

$$\begin{array}{|cccc}
\vdots & \vdots & \vdots & \iddots \\
H^0(\mathscr{T}^2_{X/Y}) & H^1(\mathscr{T}^2_{X/Y}) & H^2(\mathscr{T}^2_{X/Y}) & \ldots \\
H^0(\mathscr{T}^1_{X/Y}) & H^1(\mathscr{T}^1_{X/Y}) & H^2(\mathscr{T}^1_{X/Y}) & \ldots \\
H^0(\mathscr{T}^0_{X/Y}) & H^1(\mathscr{T}^0_{X/Y}) & H^2(\mathscr{T}^0_{X/Y}) & \ldots \\
\hline
\end{array}$$

which reduces to

$$\begin{array}{|cccc}
\vdots & \vdots & \vdots & \iddots \\
H^0(\mathscr{T}^2_{X/Y}) & H^1(\mathscr{T}^2_{X/Y}) & H^2(\mathscr{T}^2_{X/Y}) & \dots \\
H^0(\mathscr{N}_{X/Y}) & H^1(\mathscr{N}^1_{X/Y}) & H^2(\mathscr{N}_{X/Y}) & \dots \\
0 & 0 & 0 & \dots \\
\hline
\end{array}$$

From this we read off:

$$H^0(\mathscr{N}_{X/Y}) = \mathbb{T}^1_{X/Y}$$

$$0 \longrightarrow H^1(\mathscr{N}^1_{X/Y}) \longrightarrow \mathbb{T}^2_{X/Y} \longrightarrow H^0(\mathscr{T}^2_{X/Y}) \longrightarrow H^2(\mathscr{N}_{X/Y}) \longrightarrow \dots$$

This tells us a familiar thing: infinitesimal deformations of X in Y correspond precisely to global sections of the normal sheaf. The obstruction space $\mathbb{T}^2_{X/Y}$ contains the familiar $H^1(\mathscr{N}_{X/Y})$, but also something else. What can we say about $H^0(\mathscr{T}^2_{X/Y})$? As $\mathscr{N}_{X/Y} = \underline{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)$, one would guess, that $\mathscr{T}^2_{X/Y}$ should be related to $\underline{Ext}^1(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)$. In fact it usually is:

$$\mathscr{T}_0^{X/Y} = 0, \qquad \mathscr{T}_1^{X/Y} = \mathscr{I}/\mathscr{I}^2$$

and the other $\mathscr{T}_k^{X/Y}$ are concentrated at the nonregular part of the map $X \longrightarrow Y$. So if the regular part is dense, then

$$\mathscr{T}^2_{X/Y} = \underline{Ext}^1(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)\ .$$

In general there will also a term $\underline{Hom}(\mathscr{T}_2^{X/Y}, \mathscr{O}_X)$ spitting in the soup ....

We conclude that when Y is smooth, and $\mathscr{T}^2_X = 0$, then

$$\mathbb{T}^1_{X/Y} = H^0(\mathscr{N}_{X/Y})$$

$$\mathbb{T}^2_{X/Y} = H^1(\mathscr{N}_{X/Y})$$

Chapter 19

Solving the Deformation Equation

In **??** we encountered the *deformation equation*

$$\nabla\omega + \tfrac{1}{2}[\omega,\omega] = 0$$

The integrability condition for a deformed complex structure on a compact complex manifold leads to the equation of Kuranishi:

$$\overline{\partial}\theta + \tfrac{1}{2}[\theta,\theta] = 0$$

The equation $\partial \circ \partial = 0$ in the cotangent complex can be thought of as to correspond to the equation $f.r = 0$ in the resolution

$$0 \longrightarrow \mathscr{O}_X \xleftarrow{f} P^k \xleftarrow{r} P^l$$

In all these cases one ends up with the following structure; a complex (K^*, d) with the structure of a graded Lie-algebra. Usually the cohomology groups $H^0(K^*)$, $H^1(K^*)$, $H^2(K^*)$ have interpretations as infinitesimal automorphisms, infinitesimal deformations, and obstructions.

For an element $\omega \in K^1$ one can write down the equation

$$d\omega + \tfrac{1}{2}[\omega,\omega] = 0$$

Problem 19.1. Find the most general solution to this equation. This solution will represent the versal object we are looking for.

"Theorem" 19.2. *The versal solution space is isomorphic to the fibre of a map*

$$Ob : H^1(K^*) \longrightarrow H^2(K^*)\ .$$

There is an abstract "proof" of this theorem, which involves a *splitting* and an *implicit function theorem*.

The splitting. Let $Z_i = ker(K^i \longrightarrow K^{i+1})$, $B^i = Im(K^{i-1} \longrightarrow K^i)$, $H^i = Z^i/B^i$. Suppose that have splittings

$$K^i = Z^i \oplus A^i$$
$$Z^i = B^i + H^i$$

The strategy is to proof that

$$\{\omega \in H^1 + A^1 \mid d\omega + \tfrac{1}{2}[\omega,\omega] = 0\}$$

has the structure of a finite dimensional analytic space.

Of course, for this to work, we will need the Sclessinger condition (H3): $\dim H^1 < \infty$.

Using the splitting of K^2, the deformation equation splits into three parts:

$$\begin{array}{lrcl} (1) & d\omega + \frac{1}{2}\pi_{B^2}[\omega,\omega] & = & 0 \\ (2) & \frac{1}{2}\pi_{H^2}[\omega,\omega] & = & 0 \\ (3) & \frac{1}{2}\pi_{A^2}[\omega,\omega] & = & 0 \end{array}$$

Here $\pi_V : K^1 \longrightarrow V$ is the projection on a subspace V (note that $d\omega \in B^2$).

As to the *Implicit Function Theorem*, it is well known that it does not hold for in general for infinite dimensional linear spaces. For each specific deformation problem one has to put a suitable analytic structure on the K^i.

The left-hand side of equation (1) defines a map $D\colon K^1 \to B^2$, whose linearisation (=derivative) at at the origin is d. Using an implicit function theorem, one would get an isomorphism F from Z^1 from a neighbourhood U of the origin in Z^1 onto a neighbourhood of the origin in the solution set of (1).

Define

$$S = \{\, \omega \in H^1 \cap U \mid \pi_{H^2}[\omega,\omega] = 0 \,\}\ .$$

The space S is contained in the finite-dimensional vector space H^1, and this gives the complex structure we are after.

The third equation gives no further conditions, because for small ω it follows from the first two. To see this, we remember that d maps A^2 isomorphically to B^3, and compute $d\pi_{A^2}[\omega,\omega] = d[\omega,\omega] = 2[d\omega,\omega]$ by the compatibility of d and the bracket. By (1) this again is equal to $-[\pi_{B^2}[\omega,\omega],\omega] = [\pi_{A^2}[\omega,\omega],\omega]$, where we use (2) and the decomposition $1 = \pi_{B^2} + \pi_{H^2} + \pi_{A^2}$. As $d|_{A^2}$ is invertible we have, writing ψ for $\pi_{A^2}[\omega,\omega]$, the equation $\psi = (d|_{A^2})^{-1}[\psi,\omega]$ and by continuity there is a constant C, independent of ψ and ω, such that:

$$||\psi|| \leq C||\psi||\,||\omega||.$$

Therefore, for $||\omega||$ small enough we have $||\psi|| < ||\psi||$, or $\pi_{A^2}[\omega,\omega] = 0$.

In several situations this strategy was made to work, e.g. for compact complex manifolds by Kuranishi (in his second proof) and for compact complex spaces by Palamodov.

The existence of analytically versal deformations is known for

- compact complex manifolds:
  Kuranishi gave two proofs.
- isolated singularities:
  the first proof is due to Grauert, using power series methods. The proof of Pourcin uses Banach-analytc techniques.
- compact complex spaces:
  proofs by Grauert en Palamodov.
- vector bundles/ sheaves on a fixed complex space.
- The most general results are obtained by Bingener developping Palamodov's techniques further. As application he proves the case of deformations of $\pi\colon(\widetilde{X},E) \to (X,p)$ where p is a point modification with exceptional set E.

The proofs are of no use if one wants to compute versal deformations. Once the existence is established it suffices to compute formally: we quote the following useful result:

Theorem 19.3. *Suppose a versal deformation exists. Let $X \longrightarrow S$ be a family which is formally versal. Then it is versal. Moreover, a miniversal object exists.*

Some of the proofs mentioned above also prove openness of versality. The above theorem shows that the folowing weak form suffices, which is known in all the cases above.

Principle 19.4 (Openness of versality). *Let $\pi : X \longrightarrow S$ be a map. The set of points $s \in S$ where π is formally versal is Zariski-open.*

Power series Ansatz

We start with a one parameter solution of the deformation equation which we develop in a power series. We write

$$\omega = t\omega_1 + t^2\omega_2 + t^3\omega_3 + \dots \tag{4}$$

In this formula t is a parameter, which we use in a naive sense. We substitute this expression in the deformation equation:

$$t\, d\omega_1 + t^2\, d\omega_2 + \dots + \tfrac{1}{2}[t\omega_1 + t^2\omega_2 + \dots, t\omega_1 + t^2\omega_2 + \dots] = 0$$

Collecting powers of t we find the equations:

$$\begin{aligned} 0 &= d\,\omega_1 \\ 0 &= d\,\omega_2 + \tfrac{1}{2}[\omega_1, \omega_1] \\ 0 &= d\,\omega_3 + [\omega_1, \omega_2] \\ &\ \vdots \\ 0 &= d\,\omega_n + \tfrac{1}{2}\textstyle\sum_{i=1}^{n-1}[\omega_i, \omega_{n-i}]. \end{aligned}$$

The first equation states that ω_1 is a cocycle, in accordance with the fact that the equivalence classes of first order infinitesimal deformations are given by $H^1(K^*)$. The second equation gives the primary obstruction: the condition for extending ω_1 is that the cocycle $[\omega_1, \omega_1]$ is a coboundary; in other words, if the cohomology class of $[\omega_1, \omega_1]$ in $H^2(K^*)$ is zero, one can find a ω_2, which is determined up to cocycles. The secondary obstruction is only defined, if ω_2 can be found; we can still change the specific choice of ω_2, giving an indeterminacy, characteristic of Massey triple products.

This procedure tries to find a curve in the solution space, and the higher-order obstructions depend on the choices made in earlier steps. Instead we shall try to find the 'general' solution by a multivariable power series Ansatz. We should clarify the meaning of 'general' solution. The best way to do that uses the categorical language of formal deformation theory, but we do not go into this now.

Let $\dim H^1(K^*) = \tau$ (τ because for isolated complete intersection singularities the dimension of T^1 is calle dthe Tyurina number), and choose representatives $\omega_1, \dots, \omega_\tau \in C^1(K^*)$ of a basis, where $C^1(K^*)$ is a fixed complement to the 1-coboundaries $B^1(K^*) \subset K^1$. Let $t = (t_1, \dots, t_\tau)$ be the corresponding coordinates. We construct the local ring S of the solution space as quotient of $\mathbb{C}[[t]]$; let $\mathfrak{m}_\tau$ be its maximal ideal. Over $S_1 := \mathbb{C}[[t]]/\mathfrak{m}_\tau^2$ we have the solution $\sum t_i\omega_i$. To find the higher order terms, we write, similarly to (4):

$$\omega = \sum_{|\alpha| \geq 1} t^\alpha \omega_\alpha\,,$$

where this time we use a multivariable power series, and multi-index notation, so $t^\alpha = t_1^{\alpha_1} \cdots t_\tau^{\alpha_\tau}$. The primary obstruction comes from:

$$\sum_{|\alpha|=2} t^\alpha\, d\omega_\alpha + \tfrac{1}{2} \sum_{|i|=|j|=1} t^i t^j\, [\omega_i, \omega_j] = 0. \tag{5}$$

We can express the class of $[\omega_i, \omega_j]$ in $H^2(K^*)$ in terms of a basis $\Omega_1, \dots, \Omega_s$ as

$$cl([\omega_i, \omega_j]) = \sum_k c_{ij}^k \Omega_k \ .$$

The equation (5) is solvable if and only if

$$g_2^{(k)} := \tfrac{1}{2} \sum_{|i|=|j|=1} c_{ij}^k \, t^i t^j = 0 \, , \qquad \text{for } k = 1, \dots, s.$$

It is possible that some (or all) $g_2^{(k)}$ are zero, even if $\dim H^2(K^*) > 0$. Set

$$S_2 = \mathbb{C}[[t]]/(g_2) + \mathfrak{m}_\tau^3$$

and choose a basis B_2 of monomials for $\mathfrak{m}_\tau^2/(g_2) + \mathfrak{m}_\tau^3$; this can be done with a standard basis of the ideal (g_2). We will denote the set of exponents of these monomials also with B_2. Over S_2 we can solve (5): there are $\omega_\alpha \in C^1(K^*)$, with $\alpha \in B_2$, such that

$$\sum_{\alpha \in B2} t^\alpha \, d\omega_\alpha + \tfrac{1}{2} \sum_{|i|=|j|=1} t^i t^j \, [\omega_i, \omega_j] \equiv 0 \pmod{g_2}.$$

The ω_α are not unique, but determined up to elements $\psi_\alpha \in C^1(K^*)$ with $d\psi_\alpha = 0$. The possible lifts form a homogeneous space under $H^1(K^*)$. For the next step we have to solve the equation:

$$\text{(6)} \qquad \sum_{|\alpha|=3} t^\alpha \, d\omega_\alpha + \sum_{\substack{|i|=1 \\ \alpha \in B_2}} t^i t^\alpha \, [\omega_i, \omega_\alpha] \equiv 0 \pmod{g_2}.$$

Note that although the ideal (g_2) is defined in $\mathbb{C}[[t]]/\mathfrak{m}_\tau^3$, the ideal $\mathfrak{m}_\tau(g_2) \subset \mathfrak{m}_\tau^3/\mathfrak{m}_\tau^4$ is well-defined and does not depend on the extension of (g_2) to $\mathbb{C}[[t]]/\mathfrak{m}_\tau^4$. By computing the class of $[\omega_i, \omega_\alpha]$ in $H^2(K^*)$ we have again messy *Massey products*. Write $cl([\omega_i, \omega_\alpha]) = \sum_k c_{i\alpha}^k \Omega_k$. This gives:

$$g_3^{(k)} = \sum_{\substack{|i|=1 \\ \alpha \in B_2}} c_{i\alpha}^k t^i t^\alpha,$$

which defines the extension of (g_2).

Claim 19.5. *The equations $g_2^{(k)} + g_3^{(k)}$, $k = 1, \dots, s$, are the equations of the versal solution space up to third order.*

This means that we can solve (6) over $\mathbb{C}[[t]]/(g_2 + g_3) + \mathfrak{m}_\tau^4$. One continues in this way. The problem with a power series Ansatz is that the process may never end. The computation will however always be finite, if our problem is graded, and we only consider deformations of negative degree. The convention is here that a deformation corresponds to a $\partial/\partial t_i$, so the parameters t_i have positive weight, and therefore we have a bound on the possible exponents α. This means that we just with polynomials of a fixed total degree in space and deformation parameters.

Chapter 20

Computation for hypersurfaces

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d, defined by some homogeneous polynomial $f \in k[x_0, \ldots, x_n]$.

Problem 20.1. What is the dimension of $\mathbb{T}_X$?

With the same methods this problem can be solved more generally for X a complete intersection in $\mathbb{P}^n$.

Note also that no assumptions are made on the type of singularities of X.

The basis of the calculation is the global Zariski-Jacobi exact sequence for an embedding $X \subset Y$:

$$0 \longrightarrow \mathbb{T}^0_X \longrightarrow \mathbb{T}^0_Y(\mathscr{O}_X) \longrightarrow \mathbb{T}^1_{X/Y} \longrightarrow \mathbb{T}^1_X \longrightarrow \mathbb{T}^1_Y(\mathscr{O}_X) \longrightarrow \mathbb{T}^2_{X/Y}$$

As Y is smooth and X, being a hypersurface, has only locally complete intersection singularities we have by **??**

$$\mathbb{T}^{i+1}_{X/Y} = H^i(\mathscr{N}_{X/Y}) \qquad i \leq 0 \; .$$

The normal sheaf $\mathscr{N}_{X/Y}$ is of course $\mathscr{O}_X(d)$ which occurs in the exact sequence defining X:

$$0 \longrightarrow \mathscr{O} \xrightarrow{f} \mathscr{O}(d) \longrightarrow \mathscr{O}_X(d) \longrightarrow 0 \; .$$

We are going to use its long exact cohomology sequence, so we need to know the $H^i(\mathscr{O}(k))$.

Result.

$$i = 0: \qquad \dim H^0(\mathscr{O}(k)) = \binom{n+k}{k}$$

and by Serre duality:

$$i = n: \qquad H^n(\mathscr{O}(k)) \cong \left(H^0(\mathscr{O}(-n-1-k))\right)^*$$

All other groups $H^i(\mathscr{O}(k))$ vanish.

The number $\binom{n+k}{k}$ is the number of monomials of degree k in $x_0, \ldots, x_n$. We count them as follows: given a monomial $x_0^{\alpha_0} \cdots x_n \alpha_n$ write

$$\underbrace{\times \ldots \times}_{\alpha_0} \mathrm{o} \underbrace{\times \ldots \times}_{\alpha_1} \mathrm{o} \ldots \mathrm{o} \underbrace{\times \ldots \times}_{\alpha_n}$$

We have now written $k+n=\alpha_0+\cdots\alpha_n+n$ symbols. Conversely any choose of positions for the n symbols o determines the monomial so there are $\binom{n+k}{n}$ different monomials.

Using the long exact sequence

$$0 \longrightarrow H^0(\mathscr{O}(d)) \longrightarrow H^0(\mathscr{O}_X(d)) \longrightarrow H^1(\mathscr{O}) \longrightarrow$$

we see that $H^k(\mathscr{O}_X(d))=0$ for $k\geq 1$ while $\dim H^0(\mathscr{O}_X(d))=\binom{n+d}{d}-1$. Therefore

$$\dim \mathbb{T}^{i+1}_{X/Y}=\begin{cases}\binom{n+d}{d}-1, & \text{for } i=0,\\ 0, & \text{for } i>0.\end{cases}$$

So now we concentrate on $\mathbb{T}^0_Y(\mathscr{O}_X)$ and $\mathbb{T}^1_Y(\mathscr{O}_X)$. One has

$$\mathbb{T}^i_Y(\mathscr{O}_X)=H^i(\Theta\otimes\mathscr{O}_X)\ .$$

If one sees the tangent sheaf one has to use the Euler sequence

$$0\longrightarrow\mathscr{O}\longrightarrow\mathscr{O}(1)^{\oplus n+1}\longrightarrow\Theta\longrightarrow 0\ .$$

In its dual form it reads

$$0\longrightarrow\Omega\longrightarrow\mathscr{O}(-1)^{\oplus n+1}\longrightarrow\mathscr{O}\longrightarrow 0\ .$$

Let $e_0,\,\ldots,\,e_n$ be a basis of $\mathscr{O}(-1)^{\oplus n+1}$. The first map is given by

$$d(\frac{x_i}{x_j})\mapsto\frac{x_je_i-x_ie_j}{x_j^2}$$

and the second one by $(e_0,\ldots,e_n)\mapsto\sum x_ie_i$. Tensor the exact sequence

$$0\longrightarrow\mathscr{O}(-d)\xrightarrow{f}\mathscr{O}\longrightarrow\mathscr{O}_X\longrightarrow 0$$

with the locally free sheaf Θ to obtain

$$0\longrightarrow\Theta(-d)\longrightarrow\Theta\longrightarrow\Theta\otimes\mathscr{O}_X\longrightarrow 0\ .$$

We first look at $H^1(\Theta\otimes\mathscr{O}_X)$ in the long exact sequence

$$H^1(\Theta)\longrightarrow H^1(\Theta\otimes\mathscr{O}_X)\longrightarrow H^2(\Theta(-d))\longrightarrow H^2(\Theta)\ .$$

We claim that $H^1(\Theta)=\mathbb{H}^2(\Theta)=0$. This follows from the Euler sequence:

$$H^1(\mathscr{O}(1)^{\oplus n+1})\longrightarrow H^1(\Theta)\longrightarrow H^2(\mathscr{O})\longrightarrow\ldots$$

with the fact that $H^i(\mathscr{O})=0$ and $H^i(\mathscr{O}(1))=0$ for $i\geq 1$. By the way, this shows that $\mathbb{P}^n$ is rigid. We even have that $H^i(\Theta)=0$ for $i\geq 1$.

We know now that $H^1(\Theta\otimes\mathscr{O}_X)=H^2(\Theta(-d))$. To compute this last group, we twist the Euler sequence by $\mathscr{O}(-d)$:

$$0\longrightarrow\mathscr{O}(-d)\longrightarrow\mathscr{O}(1-d)^{\oplus n+1}\longrightarrow\Theta(-d)\longrightarrow 0$$

and look at its long exact sequence:

$$H^2(\mathscr{O}(1-d))^{n+1}\longrightarrow H^2(\Theta(-d))\longrightarrow H^3(\mathscr{O}(-d))\longrightarrow H^3(\mathscr{O}(1-d))^{n+1}$$

and conclude that $H^2(\Theta(-d))=0$ if $n\neq 2,3$. This means that $\mathbb{T}^1_{X/Y}\twoheadrightarrow\mathbb{T}^1_X$ and we have:

Proposition 20.2. *All deformations of a hypersurface in $\mathbb{P}^n$ are obtained by just perturbing the equation if $n\neq 2,3$.*

If $n = 3$ we look at the Serre dual and get

$$0 \longleftarrow \big(H^2(\Theta(-d))\big)^* \longleftarrow H^0(\mathscr{O}(d-4)) \longrightarrow H^0(\mathscr{O}(d-5))^4 \ .$$

If $d = 4$ then $H^0(\mathscr{O}(d-5)) = 0$ and $\dim H^2(\Theta(-4)) = 1$. The case of quartic surfaces is the $K3$-case. There is an 19-dimensional family of embedded deformations, whilst the dimenion of T^1_X is 20. The missing one is $\dim H^2(\Theta(-4))$.

If $d > 4$ the multiplication map $H^0(\mathscr{O}(d-5))^4 \longrightarrow H^0(\mathscr{O}(d-4))$ given by $(\varphi_0, \varphi_1, \varphi_2, \varphi_3) \mapsto \sum x_i\varphi_i$ is surjective so $H^2(\Theta(-d)) = 0$.

Theorem 20.3. *All deformations of hypersurfaces in $\mathbb{P}^n$ are embedded if $n \geq 4$ or $n = 3$ and $d \neq 4$.*

Plane curves

Of course for a curve to be plane is a very special property for that curve.

Now we look at the sequence

$$H^2(\mathscr{O}(-d)) \longrightarrow H^2(\mathscr{O}(1-d)^3) \longrightarrow H^2(\Theta(-d)) \longrightarrow 0$$

or in its Serre dual form

$$0 \longrightarrow H^2(\Theta(-d))^* \longrightarrow H^0(\mathscr{O}(d-4))^3 \xrightarrow{\Phi} H^0(\mathscr{O}(d-3))$$

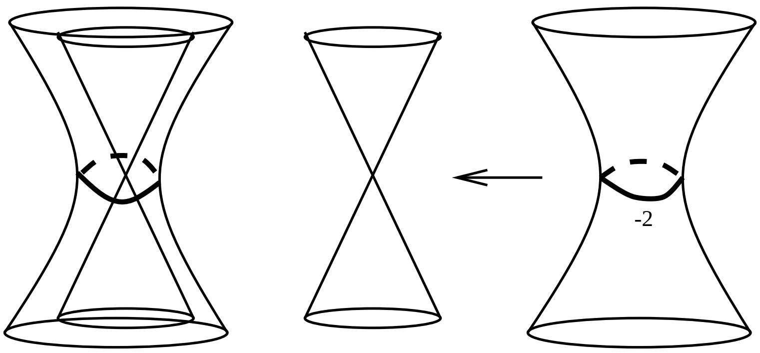
with $\Phi\colon (\varphi_0, \varphi_1, \varphi_2) \mapsto x_0\varphi_0 + x_1\varphi_1 + x_2\varphi_2$. In general, for $d \geq 5$ the kernel of Φ is huge.

1. Compute $\dim H^2(\Theta(-d))$ and find $\dim H^1(\Theta_X)$. Note that the answer $(3g-3)$ holds also for singular plane curves.

Chapter 21

Simultaneous Resolution

Let us take a look at the A_1-surface singularity.



Vanishing Cycle $\qquad$ A_1 Singularity $\qquad$ (-2) Curve

We can deform the A_1-singularity $x^2 + y^2 + z^2 = 0$ to $x^2 + y^2 + z^2 = s$. The fibre X_s is a smooth hyperboloid. As s goes to zero, a 2-sphere gets contracted to the singular point. On the other hand we can *resolve* the singularity by replacing the singular point by an exceptional curve E, a copy of $\mathbb{P}^1$, with self intersection -2. These two smooth spaces do not only look the same in the picture, they are in fact diffeomorphic. This phenomenon is not unique to the A_1-singularity. We will explain that this phenomenon characterises the $A - D - E$ singularities.

Definition 21.1. Let (X, p) be an isolated singularity.

$$\pi : (\widetilde{X}, E) \longrightarrow (X, p)$$

is called a *resolution* if

- $\widetilde{X}$ is a smooth complex space
- The map π is proper
- The map π induces an isomorphism outside E:

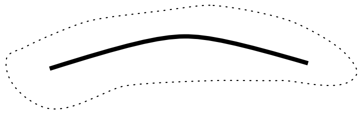
$$\widetilde{X} \setminus E \xrightarrow{\pi} X \setminus \{p\}$$

- $\operatorname{codim}_{\widetilde{X}}(E) \geq 1$

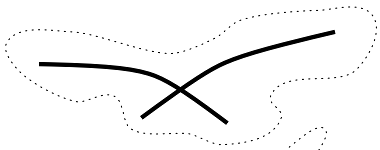
The set E is called the exceptional set of the map π, as E is constracted to the singular point p by π. When we understand a singularity really as a germ, rather than as a representative, then the manifold $\widetilde{X}$ should be considered as a germ of a manifold along E.

Any singularity can be resolved. If (X, p) is a surface singularity, there exists a unique minimal resolution. The $A - D - E$ singularities have miinimal resolutions, whose exceptional sets E consist of a union of curves E_i, each of which is isomorphic to $\mathbb{P}^1$. These E_1 intersect in the way indicated below:

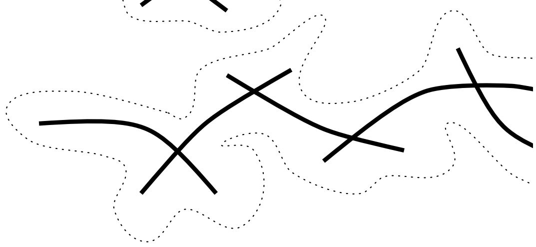
A_1



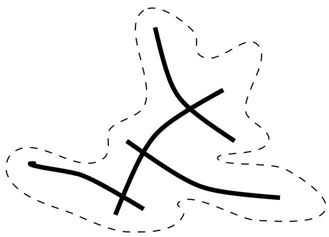
A_2



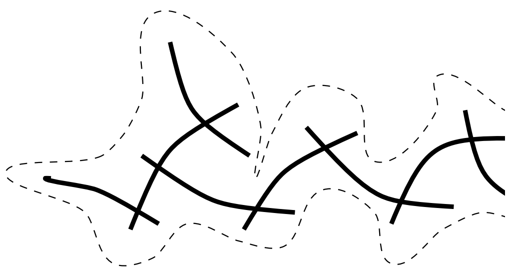
A_5



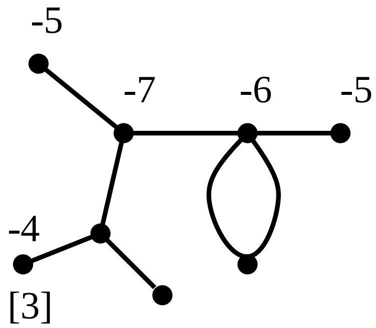
D_4



D_8



To encode the combinatrics, one usually writes down *dual graphs*, whose vertices correspond to irreducible exceptional curves and for each point of intersection between two curves there is an edge connecting the corresponding vertices. For an $A - D - E$ singularity one obtains in this way the well known Dynkin diagram with the same name. These singularities are very special and we do not want to give the impression that these are in some sense all. A generic dual graph might look as follows



Here the numbers like -7, -8, etc, indicate the self intersection of the corresponding exceptional curve. The [3] below a dot means that the corresponding curve has genus three. There can be loops in the graph, even more than one edge between vertices, indicating that the corresponding curves intersect more than once. It is standard practice not to write self intersection if it is -2, and not write the genus if it is zero. There is one necessary and sufficient condition for such a graph to occur as resolution graph of some singularity: the matrix $(E_i \cdot E_j)$ should be negative definite. This is a theorem of Grauert.

Exercise in graph theory. $A - D - E$ graphs are the only negative definite graphs with only -2 dots.

Definition 21.2. A surface singularity (X, p) is called *rational* if the scheme theoretic inverse image of p has arithmetic genus zero.

Let $\pi\colon \widetilde{X} \to X$ be a resolution of a normal surface singularity. What is the relation between deformations of $\widetilde{X}$ and of X?

Suppose $\widetilde{X}_S \longrightarrow S$ is a 1-parameter deformation of $\widetilde{X}$. Note that if (X, p) is a normal surface singularity, then one can reconstruct the local ring of the singularity by taking global section on the resolution:

$$H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = H^0(X, \mathscr{O}_X)$$

Let us try to do this in a family. One obtains a space $Y_S \longrightarrow S$ taking $H^0(\widetilde{X}_S, \mathscr{O}_{\widetilde{X}_S})$ as structure ring. This is called the *Remmert-reduction.* The map to S factors over Y_S:

$$\widetilde{X}_S \longrightarrow Y_S \longrightarrow S$$

If the special fibre Y_0 is isomorphic to X, then one gets in this way a deformation of X.

Theorem 21.3. *The fibre Y_0 is isomorphic to X if and only if*

$$H^1(\widetilde{X}_S, \mathscr{O}_{\widetilde{X}_S}) \quad \textit{is } S\textit{-flat.}$$

So we get a flat deformation of X if the (upper semi-contiuous) function

$$s \mapsto H^1(\widetilde{X}_s, \mathscr{O}_{X_s})$$

is constant. For a rational singularity one has $H^1(\mathscr{O}_{\widetilde{X}}) = 0$, and hence this condition of constancy is always fulfilled.

We are going to deform $\widetilde{X}$. As it is a smooth space we have to look at the cohomology of $\Theta_{\widetilde{X}}$:

$$\dim H^1(\Theta_{\widetilde{X}}) = ? \qquad H^2(\Theta_{\widetilde{X}}) = 0$$

We conclude that the versal base space is smooth.

Let E_i be an irreducible exceptional curve. There is a surjection

$$\Theta_{\widetilde{X}} \longrightarrow \mathscr{N}_{E_i} \longrightarrow 0$$

Define the rank 2 bundle $\mathscr{S}$ of logarithmic vectorfields by the exact sequence

$$0 \longrightarrow \mathscr{S} \longrightarrow \Theta_{\widetilde{X}} \longrightarrow \oplus \mathscr{N}_{E_i} \longrightarrow 0$$

Alternative notation: $\Theta(logE)$, as it is dual to sheaf $\Omega^1(logE)$ of logarithmic 1-forms. Locally, near the intersection of two curves, $\mathscr{S}$ is generated by $x\partial_x$ and $y\partial_y$. Easy estimate for $H^1(\Theta_{\widetilde{X}})$: we have a surjection

$$H^1(\Theta_{\widetilde{X}}) \longrightarrow H^1(\mathscr{N}_{E_i}) \longrightarrow 0$$

as *any* $H^2(coherent) = 0$ on $\widetilde{X}$. If $E_i \approx \mathbb{P}^1, E_i^2 = -2$ then $\mathscr{N}_{E_i} = \mathscr{O}_{\mathbb{P}^1}(-2)$. Hence

$$H^1(\mathscr{N}_{E_i}) = \mathbb{C}$$

For $A - D - E$ we get

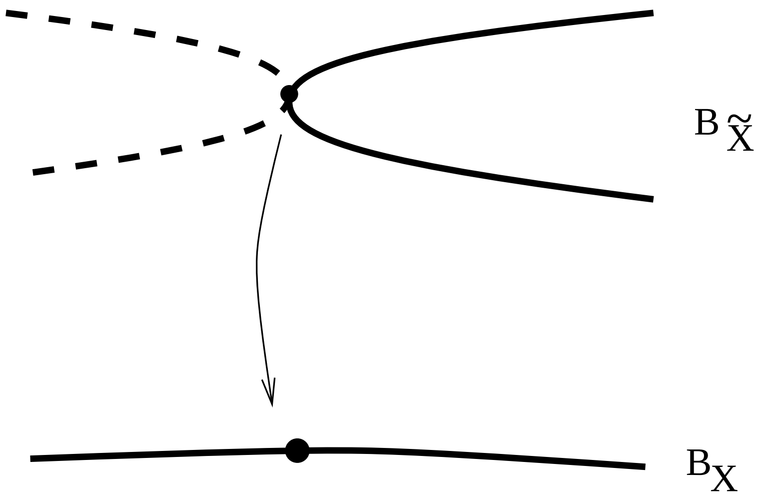
$$h^1(\Theta_{\widetilde{X}}) \geq \text{number of curves in resolution}$$

For A_k, D_k E_k in fact equality holds, and $h^1(\Theta)$ is just k. Recall from **??** that for X of type $A-D-E$ this is also the dimension of T^1_X.

So we see that for general rational surface singularities one gets a map from the versal base space $B_{\widetilde{X}}$ of $\widetilde{X}$ to that of X

$$B_{\widetilde{X}} \longrightarrow B_X$$

For X of type $A - D - E$ both spaces have dimension k. What is this map? For $X = A_1$ it a map between two smooth 1-dimensional germ. We will see that $B_{\widetilde{X}} \longrightarrow B_X$ is the *squaring map* $t \mapsto s = t^2$.



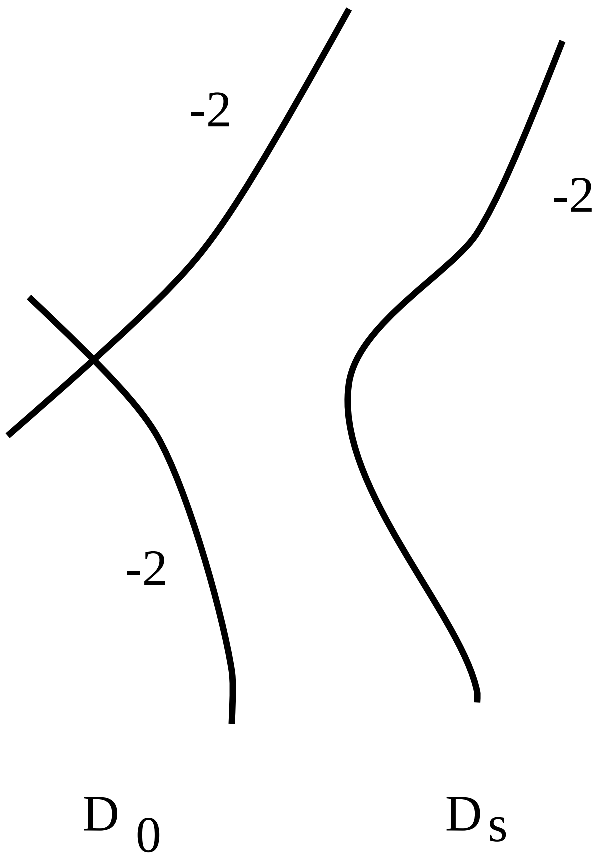
So the map $\mathbb{T}^1_{\widetilde{X}} \longrightarrow T^1_X$ on the level of tangent spaces is the zero map.

For the A_1-singularity one can see the map from the theory of flops for threefolds. The singularity $xy - z^2 + t = 0$ becomes after squaring isomorphic to $Y\colon xy - uv = 0$ (set $z + t = u$, $z - t = v$). A resolution $\widetilde{Y}$ of Y is obtained by closing the graph of the function $x/u\colon Y \to \mathbb{P}^1$. The exceptional set E is one-dimensional (just the $\mathbb{P}^1$ above the origin), so this is a so-called small resolution. An other choice is to take the function $x/v = y/u$. In the original coordinates this is the choice between $z + t$

and $z - t$. The function t has as zero fibre on $\widetilde{Y}$ a resolution of X. Both possible deformations are not the same but they induce the same deformation of X. The same picture holds more or less for the other *A*–*D*–*E* singularities.

To study the deformation of the resolution we look at the inverse image of the discriminant in the base space of X, so at points for which the fibre blows down to one or more singularities.

Consider an effective divisor $D \subset \widetilde{X}$, and suppose we can lift D to a relative divisor $D_S \subset \widetilde{X}_S$. The number of irreducible components of the divisor $D_s \subset \widetilde{X}_s$ can change, but the self-intersection $(D_s \cdot D_s)$ is constant.



Suppose that D_s is irreducible and reduced for $s \neq 0$. Then $h^0(\mathscr{O}_{D_s}) = 1$. To what sort of divisors on $\widetilde{X}$ can such an D_s specialize? For a rational surface singularity, $0 = H^1(\mathscr{O}_{\widetilde{X}}) \longrightarrow H^1(\mathscr{O}_D) \longrightarrow 0$. So $1 = \chi(\mathscr{O}_D) = -\frac{1}{2}D(D-K)$. If $E \approx \mathbb{P}^1$ and $E^2 = -2$, then $-2 = E(E-K)$, so $K \cdot E = 0$, from which we see that for X of type $A - D - E$ one has $K = 0$. Hence, the divisors with $\chi(\mathscr{O}_D) = 1$ we were looking for are those with $D^2 = -2$. It is a question of graph theory/combinatorics to find all such divisors. The elements D in the $A - D - E$ lattice $\oplus_{i=1}^k \mathbb{Z}[E_i]$ with $D \cdot D = -2$ are called the roots of the root system. The Weyl group is generated by reflections in these roots.

Let us look at deformations over which a given root D lifts. We deform $D \to \widetilde{X}$ so we have the exact sequence of deformation functors

$$\mathrm{Def}_{D/\widetilde{X}} \longrightarrow \mathrm{Def}_{D\to\widetilde{X}} \longrightarrow \mathrm{Def}_{\widetilde{X}}$$

In the long exact sequence for the $\mathbb{T}^i$ we have $\mathbb{T}^{I+1} = H^i(\mathscr{N}_D)$, and as D is a curve of arithmetic genus zero and $D^2 = -2$, $\mathscr{N}_D = \mathscr{O}_D(-2)$ and $H^0(\mathscr{N}_D) = 0$, while $H^1(\mathscr{N}_D) = \mathbb{C}$. We obtain therefore

$$\begin{array}{ccccccccccc} 0 = \mathbb{T}^1_{D/\widetilde{X}} & \longrightarrow & \mathbb{T}^1_{D\to\widetilde{X}} & \longrightarrow & \mathbb{T}^1_{\widetilde{X}} & \longrightarrow & \mathbb{T}^2_{D/\widetilde{X}} & \longrightarrow & \mathbb{T}^2_{D\to\widetilde{X}} & \longrightarrow & \mathbb{T}^2_{\widetilde{X}} = 0 \\ & & & & \| & & \| & & & & \\ & & & & H^1(\Theta_{\widetilde{X}}) & \longrightarrow & H^1(\mathscr{N}_D) & & & & \end{array}$$

One can even identify $\mathbb{T}^2_{D\to\widetilde{X}}$ with the H^2 of a coherent sheaf on $\widetilde{X}$ so in fact the map to $H^1(\mathscr{N}_D)$ is surjective.

Conclusion. There is a codimension one subspace $B_{D\to\widetilde{X}} \subset B_{\widetilde{X}}$ over which the root D lifts to $\widetilde{X}$.

At a general point of $B_{D\to\widetilde{X}}$ the curve D is smooth and blows down to an A_1 singularity. Locally there the map to B_X is the squaring map.

We have a diagram

$$\begin{array}{ccc} B_{\widetilde{X}} & \longrightarrow & B_X \\ \Big\uparrow & & \Big\uparrow \\ B_{D\to\widetilde{X}} & \longrightarrow & \Delta \end{array}$$

where Δ is the discriminant, the locus of non smooth fibres.

Theorem 21.4.

$$B_X \cong B_{\widetilde{X}}/W$$

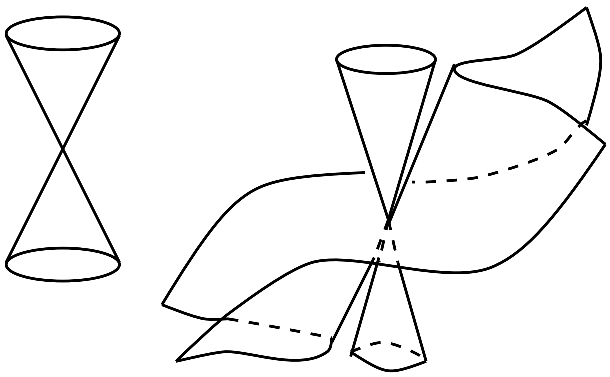
where W is the Weyl group of the appropriate type, which acts on $B_{\widetilde{X}}$ by reflections in the hyperplanes $B_{D\to\widetilde{X}}$.

Chapter 22

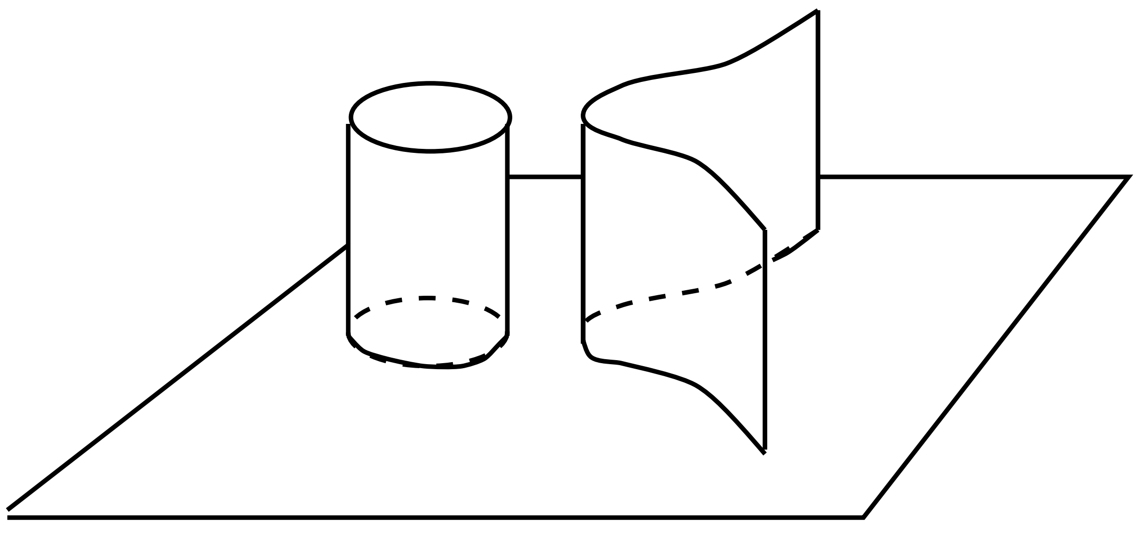
Cubic Surfaces

Let X be a surface singularity and $\widetilde{X}$ its resolution. One can ask the question whether deformations of X correspond to deformations of $\widetilde{X}$ and vice versa. In general this is not the case.

Example 22.1. Consider the hypersurface singularity $\widetilde{E}_6$ in $(\mathbb{C}^3, 0)$ given by $x^3 + y^3 + z^3 = 0$. This is not an A–D–E singularity: it has multiplicity 3 and it is the cone over a smooth elliptic curve in $\mathbb{P}^2$, just as A_1 is the cone over a smooth conic in $\mathbb{P}^2$.



How to get a resolution of X? The answer is: blow up $\mathbb{C}^3$ in the point $\{0\}$. The picture now looks like:



The exceptional curve is the elliptic curve we started with. One way to deform $\widetilde{X}$ is by changing the structure of the elliptic curve. This are the deformations which blow down. The singularity is not rational, in fact $\dim H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 1$. Other deformations of $\widetilde{X}$ do exist but they change $H^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$. So we do not get many deformations of X by deforming just $\widetilde{X}$.

Of course X itself has a lot of deformations: $\dim T^1_X = 8$ and a basis of T^1_X consists of the monomials

$$1, x, y, z, yz, xz, xy, xyz.$$

The last monomial gives the deformation $x^3 + y^3 + z^3 + txyz$ which means changing the elliptic curve. The versal deformation is given by the formula

$$x^3 + y^3 + z^3 + t_0 + t_1 x + t_2 y + t_3 z + t_4 yz + t_5 xz + t_6 xy + t_7 xyz\ .$$

A fibre of this deformation will be an affine cubic surface. Therefore the following questions are equivalent:

$$\begin{gathered}\text{which singularities can appear in a fibre of a deformation of } \widetilde{E}_6 \text{ ?}\\ \Updownarrow \\ \text{which singularities can appear on a cubic surface in } \mathbb{P}^3 \text{ ?}\end{gathered}$$

We can generalise the question and pose the

Problem 22.2. What kind of isolated singularities can appear on a projective surface of degree d in $\mathbb{P}^3$?

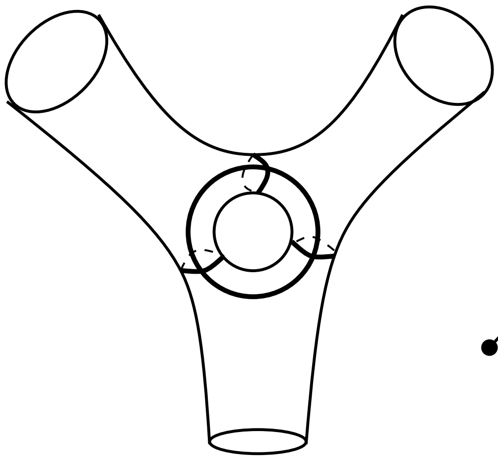
A general upper bound for the number of singularities of specified types in all dimensions is provided by Varchenko's estimate [A-G-V] whereas for A–D–E singularities on surfaces an asymptotically better estimate is available (Miyaoka-Yau). The situation for ordinary double points (A_1-singularities) on surfaces of low degrees is:

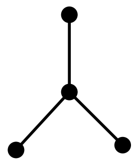
degree	$\#A_1$'s
2	1
3	4
4	16
5	31
6	65
7	unknown!

Dynkin diagram

For the A–D–E singularities we obtained the A–D–E diagram as resolution graph. By the existence of a simultaneous resolution one can equally well consider the topology of a smooth fibre and this gives the correct generalisation for non-rational singularities. So we consider the *Milnor fibre*: let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function with a singularity at the origin, take a small closed ball with center at the origin and intersect the fibre $f = t$ for t very small with the closed ball. The resulting smooth real $2n$-dimensional manifold with boundary is by definition the Milnor fibre.

Example 22.3. Consider the curve singularity $D_4\colon x^3 + y^3$. To get a better real picture of the zero set we take the real form $x^3 - xy^2$. As the singularity is quasi-homogeneous we can take a large ball and $t = 1$ so we look at the intersection of the affine part $x^3 - xy^2 = 1$ of an elliptic curve with a large ball. We get a Riemann surface F of genus one with three holes coming from the three points at infinity:

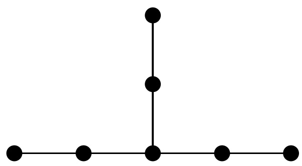




The cycles in the picture generate $H_1(F,\mathbb{Z})$ and intersect according to the D_4 graph.

Cubic surfaces

Similarly to the D_4 example we now want to look at the affine part F of a cubic surface. One can see cycles which intersect according to the $\widetilde{E}_6$ diagram:



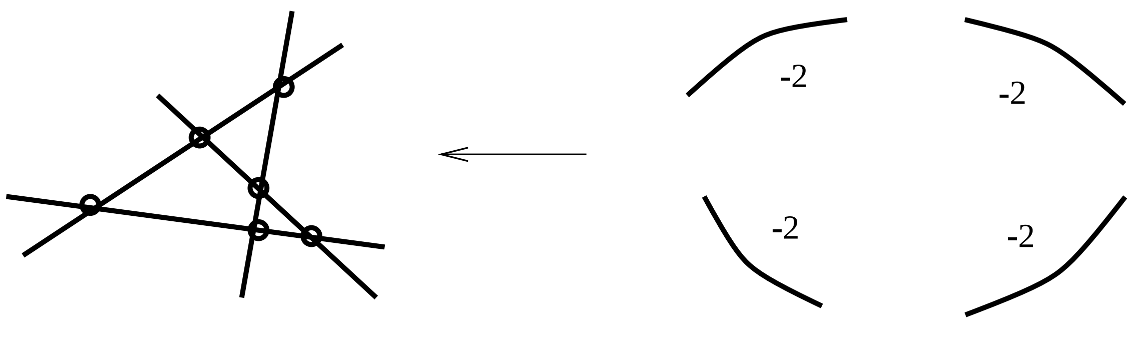
A smooth cubic surface contains 27 lines.

One obtains a cubic surface by blowing up $\mathbb{P}^2$ in six points $p_1, \dots p_6$, not all on a conic and not three on a line. The vector space of polynomials of degree 3 vanishing in $p_1, \dots p_6$ has dimension four and the choice of a basis $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3\}$ gives rise to a rational map

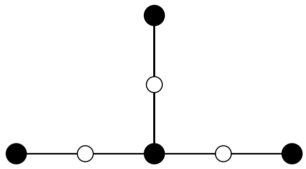
$$(\varphi_0 : \varphi_1 : \varphi_2 : \varphi_3)\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3 .$$

hier moet meer uitgelegd worden

Singular cubics are obtained by taking the six points special. For the four nodal cubic we take the six vertices of a complete quadrilateral. Upon blowing up the six vertices the strict transforms of the four sides are disjoint curves of self-intersection -2 but the polynomials of degree three intersect the lines only in the base points so the map to $\mathbb{P}^3$ blows the lines down to A_1 singularities.



The four disjoint (-2)-curves can be seen in the $\widetilde{E}_6$-diagram:



This illustrates that we get a similar game as with the A–D–E singularities.

- How to get the singularities on a cubic surface which is not the cone over an elliptic curve?
- Apply the following operation on the $\widetilde{E}_6$ diagram
  - remove some points
  - remove the edges adjacent to them
- What is left is the Dynkin diagram of (maybe several) A–D–E singularities.

Theorem 22.4. *You get them all this way!*

The explanation of this phenomenon using the deformation theory of $\widetilde{E}_6$ was done in the 70's by Eduard Looijenga.

2. Make the list.

Chapter 23

Calabi-Yau threefolds

We have seen that the basic properties of elliptic curves can be generalised in different ways to surfaces, giving tori on the one hand and $K3$-surfaces on the other. We now take the step to threefolds and study the analogues of $K3$-surfaces.

Definition 23.1. A smooth complex 3-dimensional manifold X is called a Calabi-Yau threefold if $\omega_X \cong \mathscr{O}_X$ and $H^0(\Omega^1_X) = H^0(\Omega^2_X) = 0$.

The Chern numbers are:

$$\begin{aligned} c_1 &= 1 - g \\ \frac{c_1^2 + c_2}{2} &= 1 - q + p_g \\ \frac{c_1 c_2}{24} &= 1 - h^1 + h^2 - 1 \end{aligned}$$

The motivition from physics to look at such manifolds is that our univers is not four dimensional but $U = \mathbb{R}^{1,3} \times M^6$ where M^6 is a manifold with a diameter in the order of $\varepsilon = 10^{-33}$cm so one can think of ε^2 being zero. The space M^6 should have a Ricci-flat metric which implies $c_1 = 0$. Yau proved the Calabi-conjecture that conversely Ricci-flat implies $c_1 = 0$.

For Calabi-Yau threefolds we find the following invariants. Write $a = h^1(\Omega^1)$, $b = h^2(\Omega^1)$. By Serre duality $\left(H^1(\Omega^1)\right)^* = H^1(\Theta)$ so the number b is also the number of deformation parameters, while the obstructions land in a space of dimension a, the number of divisors. The Hodge diamond looks like

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & a & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The Euler number is given by $e = 2(a - b)$. In the following table we list e and a for some classes of examples.

	example	e	a
1)	double octics	-296	1
2)	quintics in $\mathbb{P}^4$	-200	1
3)	$(2,4)$ in $\mathbb{P}^1 \times \mathbb{P}^3$	-168	2
4)	$(3,3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$		2
	$\vdots$		
N)	elliptic fibre product	0	~ 20

A double octic solid is a double cover of $\mathbb{P}^3$ branched along an octic surface. The symbol $(2,4)$ in $\mathbb{P}^1 \times \mathbb{P}^3$ means a divisor of type $(2,4)$ on the fourfold $\mathbb{P}^1 \times \mathbb{P}^3$. In general, in a fourfold X with $-K_X$ a general anticanonical divisor is a smooth Calabi-Yau.

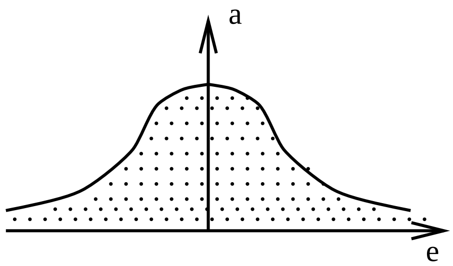
The construction of the last example starts with two elliptic surfaces E_1, E_2 with $K = -F$, given by a pencil of plane cubics. We define the threefold as fibre product of the surfaces: $E_1 \times_{\mathbb{P}^1} E_2 = \{(e-1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = p_2(e_2)\}$.

$$\begin{tikzcd} & E_1 \times_{\mathbb{P}^1} E_2 \arrow[dl] \arrow[dr] & \\ E_1 \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

For a very ample section of a fourfold the cohomology agrees up to h^2 with that of the fourfold as an immediate consequence of the exact sequence

$$0 \longrightarrow \mathscr{O}(-D) \longrightarrow \mathscr{O} \longrightarrow \mathscr{O}_D \longrightarrow 0\ .$$

This shows that $a = 1$ for quintics in $\mathbb{P}^4$. But the physicists tell us that that the distribution of the numbers (a, e) looks like:



We can get higher values for a on singular quintics, take e.g. the quintic with an equation of the form

$$x_0 F_0 + x_1 F_1 = 0.$$

It has in general 16 singular points, namely where $x_0 = F_0 = x_1 = F_1 = 0$.

What happens if we impose a node?

Let X_0 be a quintic with a node in the point P. Resolve the singularity by blowing up the point P. Locally at P the threefold is isomorphic to the cone over a smooth quadric, so the exceptional divisor is a quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[d] \\ P \arrow[r, hook] & X_0 \end{tikzcd}$$

The quadric Q in $\widetilde{X}$ can be blown down along either ruling to a threefold which is still smooth, but might not be projective. The exceptional set in the new threefold is a rational curve with normal bundle $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$. This is a so called small resolution and we get two of them fitting in the following diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr] & \\ X_1 \arrow[dr] & & X_2 \arrow[dl] \\ & X_0 & \end{tikzcd}$$

The singular quintic is a degeneration of a smooth threefold X_t and if we compare Euler numbers we see that it goes up by two: $e(X_1) = e(X_2) = e(X_t) + 2$.

So it is interesting to look at rational curves on Calabi-Yau threefolds. Let $C \cong \mathbb{P}^1$. In the normal bundle sequence

$$0 \longrightarrow \Theta_C \longrightarrow \Theta_X|_C \longrightarrow \mathscr{N}_{C/X} \longrightarrow 0$$

we have that $\Theta_C = \mathscr{O}(2)$ and $\longrightarrow \Theta_X|_C$ splits as $\mathscr{O}(a_1) \oplus \mathscr{O}(a_2) \oplus \mathscr{O}(a_3)$ with $a_1 + a_2 + a_3 = 0$, so $\deg \mathscr{N}_{C/X} = -2$ and therefore $\mathscr{N}_{C/X} = \mathscr{O}(a) \oplus \mathscr{O}(-a-2)$. In the generic case one expects that $a = -1$, as in the example coming from the small resolution. In this case both $H^0(C, \mathscr{N}_{C/X})$ and $H^1(C, \mathscr{N}_{C/X})$ vanish. This implies that "curves do not deform". Examples exist where curves do deform (there $a \neq= 1$), but on a "generic" Calabi-Yau rational curves are rigid.

Let us look at quintics that contain a line, e.g. $x_0 = x_1 = x_2 = 0$. The equation of the quintic Q has then the form

$$x_0 Q_0 + x_1 Q_1 + x_2 Q_2 = 0$$

with the Q_i quartic forms. We can compare the normal bundle of the curve in Q with that in $\mathbb{P}^4$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{N}_{\mathbb{P}^1/Q} & \longrightarrow & \mathscr{N}_{\mathbb{P}^1/\mathbb{P}^4} & \longrightarrow & \mathscr{N}_{Q/\mathbb{P}^4}|_{\mathbb{P}^1} & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & \mathscr{N}_{\mathbb{P}^1/Q} & \longrightarrow & \bigoplus_{i=0}^2 \mathscr{O}(1) & \longrightarrow & \mathscr{O}(5) & \longrightarrow & 0 \\ & & & & (x_0, x_1, x_2) & \longmapsto & \sum x_i Q_i & & \end{array}$$

This gives the exact sequence

$$0 \longrightarrow H^0(\mathscr{N}) \longrightarrow \underset{6}{\bigoplus_{i=0}^2 H^0(\mathscr{O}(1))} \longrightarrow \underset{6}{H^0(\mathscr{O}(5))} \longrightarrow H^1(\mathscr{N}) \longrightarrow 0$$

which shows that depending on the rank of the map in the middle the normal bundle can be $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$ or $\mathscr{O} \oplus \mathscr{O}(-2)$.

An example of a familiy of lines is provided by the Fermat quintic $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$ which contains the lines $\{(u, -u, av + bv + cv) \mid a^5 + b^5 + c^5 = 0\}$.

In general one expects a finite number of curves for each degree. For the generic quintic this is

d	# of curves
1	2875
2	609250
3	317206375

Chapter 24

T^1–lifting property

Let X be a smooth Calabi-Yau threefold. Then the dualising sheaf ω_X is isomorphic to $\mathscr{O}_X$ and consequently $\Theta_X \cong \Omega^2_X$. Therefore we know the dimensions of the cohomology groups which are relevant for deformation theory. Recall that the Hodge diamond looks like

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & a & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Therefore

$$\begin{aligned} \mathbb{T}^1_X &= H^1(\Theta_X) &= H^1(\Omega^2) &= \mathbb{C}^b \\ \mathbb{T}^2_X &= H^2(\Theta_X) &= H^2(\Omega^2) &= \mathbb{C}^a \end{aligned}$$

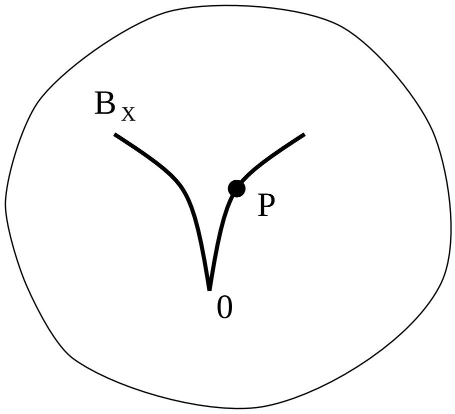
The general theory tells us that the base space B_X of the versal deformation of X is the fibre $vp^{-1}(0)$ of a map of germs

$$\varphi = \mathrm{ob}\colon (\mathbb{T}^1_X, 0) \longrightarrow (\mathbb{T}^2, 0)\ .$$

Here we are in a *lucky case*: the map φ is identically zero and the base space B_X is smooth.

Theorem 24.1 (Bogolomolov, Tian, Todorov). *If X is a smooth Calabi-Yau manifold, then the miniversal base space (also called Kuranishi space) B_X is smooth.*

Proof. Suppose that B_X is singular. Consider a nearby point P.



Then the Zariski tangent space to B_X at the point P has smaller dimension then the Zariski tangent space at 0. But $T_0 B_X = \mathbb{T}^1_X = \mathbb{C}^b$ and $T_P B_X = \mathbb{T}^1_{X_P}$ where X_P is the threefold parametrised by the point P. But also $\dim \mathbb{T}^1_{X_P} = \mathbb{C}^b$ as a and b are toplogical invariants ($\dim \mathbb{H}^2(X,\mathbb{C}) = a$ and $\dim H^3(X,\mathbb{C}) = 2b+2$) which do not change in a family. $\square$

The problem with this proof is that we might not be able to find a point P as in the picture. Consider instead a family $X_S \to S$. Then we have the group $\mathbb{T}^1_{X_S/S}$ of infinitesimal deformations of X_S over S. Let as usual $\mathbb{D} = \operatorname{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ and consider the diagram

$$\begin{array}{ccccccc} X & \hookrightarrow & X_S & \hookrightarrow & X'_S & \longleftarrow & \text{something} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \{0\} & \longrightarrow & S & \hookrightarrow & S\times\mathbb{D} & \longleftarrow & \{0\}\times\mathbb{D} \end{array}$$

In this way one obtains a natural map $\mathbb{T}^1_{X_S/S} \to \mathbb{T}^1_X$. If we take $S = \mathbb{C}\{t\}$ we can look at multiplication with t:

$$0 \longrightarrow \mathscr{O}_{X_S} \xrightarrow{.t} \mathscr{O}_{X_S} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

and the corresponding long exact sequence

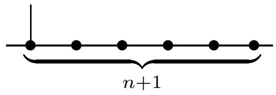
$$\mathbb{T}^1_{X_S/S} \xrightarrow{.t} \mathbb{T}^1_{X_S/S} \longrightarrow \mathbb{T}^1_X \longrightarrow \mathbb{T}^2_{X_S/S} \xrightarrow{.t} \dots$$

$\mathbb{T}^1$-lifting principle. If the maps $\mathbb{T}^1_{X_S/S} \to \mathbb{T}^1_X$ are surjective for all S then the base space B_X is smooth.

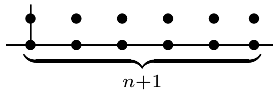
This principle was formulated by Ziv Ran. It can also be considered as an instance of a "T^2-injecting principle".

Consider any deformation functor $D(\,..)$ satisfying the Schlessinger conditions and look at three types of Artinian algebras:

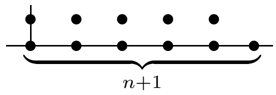
$A_n = k[t]/t^{n+1}$



$B_n = k[t, \varepsilon]/(t^{n+1}, \varepsilon^2)$



$C_n = k[t, \varepsilon]/(t^{n+1}, t^n\varepsilon, \varepsilon^2)$



where $C_n = B_{n-1} \times_{A_{n-1}} A_n$. Fix an element $X_n \in D(A_n)$. Then

$$T^1(X_n/A_n) := \{Y_n \in D(B_n) \mid Y_n|A_n = X_n\} \, .$$

Lift $X_n \in D(A_n)$ to $X_{n+1} \in D(A_{n+1})$. Together with a given $Y \in T^1(X_n/A_n)$ this lift defines a deformation over C_{n+1}, just by glueing the deformation.

$\mathbb{T}^1$-lifting principle. The complete local ring R of the versal base (the "hull") is smooth if always $T^1(X_{n+1}/A_{n+1}) \twoheadrightarrow T^1(X_n/A_n)$, i.e. $D(B_{n+1}) \twoheadrightarrow D(C_n)$.

Let R be the versal base. If

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

is a small extension of rings ($\mathfrak{m}_A I = 0$), then $X_A = j^* X_R$ for some map $j \colon R \to A$ Now consider the diagram of a lifting

$$\begin{array}{ccc} & & \uparrow \\ R & \xrightarrow{j} & A \\ & \searrow & \uparrow \\ & & A' \end{array}$$

Formal *smoothness* of R means, that any such j can be lifted to some $j' : R \longrightarrow A'$, and in this way one obtains a lift $X_{A'} = (j')^* X_R \in D(A')$. Conversely, *versality* means, that any $X_{A'} \in D(A')$ that lifts X_A can be induced by some j' that lifts j. In fact, formal smoothness is *equivalent to or defined as* the property that every map j can be lifted to a j', so the principle states that one can check this using not all A and A', but only the special rings B_{n+1} and C_n.

In order to apply the above to Calabi-Yau threefolds we first recall some results on *cohomology and base change* (see [Ha]). We start with $S = \mathbb{C}[[t]]$ and an S-flat S-module M_s. We have the exact sequence

$$0 \longrightarrow M_S \xrightarrow{\cdot t} M_S \longrightarrow M \longrightarrow 0$$

with its long exact sequence

$$\longrightarrow \ldots H^k(M_S) \xrightarrow{\cdot t} H^k(M_S) \longrightarrow H^k(M) \longrightarrow H^{k+1}(M_S) \xrightarrow{\cdot t} H^{k+1}(M_S) \longrightarrow \ldots$$

Proposition 24.2. *Assume that the $H^k(M_S)$ are finitely generated. Then*

(1) If $H^{k+1}(M) = 0$, then $H^{k+1}(M_S) = 0$ and the reduction map $H^k(M_S) \to H^k(M)$ is surjective.

(2) If moreover $H^{k-1}(M_S) \twoheadrightarrow H^{k-1}(M)$ then $H^k(M_S)$ is S-flat.

The point of *cohomology and base change* is, that these last two conclusions hold for any ring S, and not just $\mathbb{C}[[t]]$!

Consider now a flat deformation $X_S \to S$ of a Calabi-Yau threefold. We want to show that $\mathbb{T}^1_{X_S/S}$ and $\mathbb{T}^2_{X_S/S}$ are S-flat. We saw that $\mathbb{T}^1_X = H^1(\Omega^2_X)$ and $\mathbb{T}^2_X = H^2(\Omega^2_X)$. Now

$$\begin{aligned}\mathbb{T}^1_{X_S/S} &= H^1(\Omega^2_{X_S/S}) = rmHom_S(H^1(\Omega^1_{X_S/S}), S)\\ \mathbb{T}^2_{X_S/S} &= H^2(\Omega^2_{X_S/S}) = rmHom_S(H^2(\Omega^1_{X_S/S}), S)\end{aligned}$$

and $\Omega^1_{X_S/S}$ is S-flat.

Now one can prove that

$\boxed{H^1(\Omega^1_{X_S/S}) \twoheadrightarrow H^1(\Omega_X) \text{ ???}}$

Deforming Nodal Varieties

The theorem about smoothness of the base space B_X continues to hold if the threefold X with $\omega_X \cong \mathscr{O}_X$ has isolated cDV-singularities. Such singularities have small resolutions: a resolution $\widetilde{X} \to X$ with exceptional set of codimension two. This implies that $\omega_{\widetilde{X}} \cong \mathscr{O}_{\widetilde{X}}$. The simplest example is that (X, p) is an A_1-singularity. It can be resolved (in two ways) with an exceptional $\mathbb{P}^1$ with normal bundle $\mathscr{N} = \mathscr{O}(-1) \oplus \mathscr{O}(-1)$. You cannot get rid of such a curve by deforming it, as $\mathbb{T}^1_{C\to\widetilde{X}} \cong \mathbb{T}^1_{\widetilde{X}}$:

$$\begin{array}{ccccccc}\mathbb{T}^1_{C/\widetilde{X}} & \longrightarrow & \mathbb{T}^1_{C\to\widetilde{X}} & \longrightarrow & \mathbb{T}^1_{\widetilde{X}} & \longrightarrow & \mathbb{T}^2_{C/\widetilde{X}}\\ \| & & & & & & \|\\ H^0(\mathscr{N})=0 & & & & & & H^0(\mathscr{N})=0\end{array}$$

Theorem 24.3 (Friedman). *If $\widetilde{X} \to X$ is a small resolution of a threefold with isolated singularities then* $\operatorname{Def}\widetilde{X} \hookrightarrow \operatorname{Def} X$.

For a Calabi-Yau with only nodes we get the exact sequence

$$0 \longrightarrow H^1(\Theta_X) \longrightarrow \mathbb{T}^1_X \longrightarrow H^0(\mathscr{T}^1_X) \longrightarrow H^2(\Theta_X) \longrightarrow \mathbb{T}^2_X \longrightarrow 0$$

One gets

$$\mathbb{T}^2_X = H^4(\widetilde{X}, \mathbb{C})/\sum[C_i]$$

where the $[C_i]$ are the Poincaré duals of the exceptional curves C_i resolving the nodes.

Chapter 25

Deforming Calabi-Yau threefolds

Chapter 26

Exercises

3. (Perturbations of the equations of the coordinate axes) Consider the equations

$$f_1 = yz, \qquad f_2 = xz, \qquad f_3 = xy$$

with relations

$$\begin{aligned} xf_1 - yf_2 &= 0 \\ yf_2 - zf_3 &= 0 \end{aligned}$$

as in the first lecture. Deform the equations to

$$\begin{aligned} F_1 &= yz - s \\ F_2 &= xz - s \\ F_3 &= xy - s \end{aligned}$$

and try to lift the relations. (Hint: start computing $xF_1 - yF_2$). Suppose $s \neq 0$ so you may divide by s. Find in this way new generators of the ideal for (fixed) $s \neq 0$. What is the geometric interpretation?

Now take

$$\begin{aligned} G_1 &= yz + ty + tz \\ G_2 &= xz \\ G_3 &= xy \end{aligned}$$

Determine the zero locus. Lift the relations.

Let $P = \mathbb{C}[x, y, z; t]$. One has an exact sequence

$$0 \longleftarrow \mathscr{O}_{X_T} \longleftarrow P \xleftarrow{G} P^3 \xleftarrow{R} P^2$$

with G the row vector (G_1, G_2, G_3) and R the relation matrix. Write down this matrix and compute its maximal minors.

4. (Cone over the rational normal curve of degree 4). Let $\mathbb{P}^1 \to P^4$ be the embedding given by

$$z_i = s^{4-i}t^i, \qquad i = 0, \ldots, 4.$$

The equations are the minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

Relations are easy to get: double a row, say the first one:

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_0 & z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

and compute the 3×3 minors by developping them with respect to the first row. How many relations do you get this way. Generalise to the rational normal curve $\mathbb{P}^1 \to P^d$ of degree d.

Now look at the same equations in $\mathbb{C}^5$, or in other words: take the affine cone. Written out the equations are

$$\begin{array}{ccc} z_0z_2 - z_1^2 & z_1z_3 - z_1^4 & z_2z_4 - z_3^2 \\ z_0z_3 - z_1z_2 & & z_1z_4 - z_2z_3 \\ & z_0z_4 - z_1z_3 & \end{array}$$

Compute the zero locus of the three equations in the upper row. They form a complete intersection which coincides with our cone outside the coordinate hyperplanes. What can you say about a generic perturbation of these equations (smooth, irreducible?)? Do you get a deformation of the cone by such a generic perturbation?

Now consider the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 + t_1 & z_2 + t_2 & z_3 + t_3 & z_4 \end{pmatrix}$$

Relations are easy!

Consider also the 2×2 minors of

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 + s & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

For $s = 0$ you get the same ideal as before. Lift the original relations.

5. Write down a quartic curve with an A_4-singularity and draw a picture. The singularity is also called rhamphoid (= beak-like) cusp.

6. Find adjacencies $A_k \to A_{k-1}$, $D_4 \to A_3$ and $E_7 \to D_6$.

7. Find the invariants of the action of $G = \mathbb{Z}/n$ on $\mathbb{C}^2$ defined by $(x, y) \mapsto (\zeta x, \zeta y)$ with $\zeta = e^{2\pi i/n}$ a primitive nth root of unity. Determine the equations of the image of the resulting map $\mathbb{C}^2/G \to \mathbb{C}^N$.

8. Determine the image of the map

$$\varphi_S : (\mathbb{C} \coprod \mathbb{C}) \times S \longrightarrow \mathbb{C}^3 \times S$$

defined by

$$\begin{array}{cccc} (x & , s) & \mapsto & ((x, o, s), s) \\ (\quad y & , s) & \mapsto & ((0, y, s), s) \end{array}$$

and check that $\mathrm{Im}(\varphi_S)_0 \neq \mathrm{Im}\,\varphi_0$. (This is called “pulling apart two lines”.)

9. What is an equation for the 4-nodal cubic surface in $\mathbb{P}^3$?

10. Compute the Euler number of a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by a bihomogeneous polynomial $F(X_0, X_1; Y_0, Y_1, Y_2)$ of bidegree (d_1, d_2).

11. Compute K^2 of the above example.

12. How will an ordinary triple point (say on a hypersurface in $\mathbb{P}^3$) affect the pluri-genera?

13. (Deformation of maps) Two maps $f\colon X \to Y$ and $f'\colon X \to Y$ are called (left-right) equivalent if there exist automorphisms $g\colon X \to X$ and $h\colon Y \to Y$ such that $f = h \circ f' \circ g$.

A deformation of f over S is a map

$$\begin{array}{rcl} f_S\colon X \times S & \longrightarrow & Y \times S \\ (x,s) & \mapsto & (f(x,s),s) \end{array}$$

such that $f(x,0) = f(x)$.

a) When would you call two deformations over S equivalent?

b) Let $X = (\mathbb{C}, 0)$ and $Y = \{xy - z^2 = 0\} \subset (\mathbb{C}^3, 0)$ and consider the map $f\colon X \to Y$ given by $t \mapsto (t,t,t)$. Find a non-trivial deformation of f over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$.

c) Show that this deformation does not lift to second order.

14. Explain the difference between $\mathbb{C}[[s]][x]$ and $\mathbb{C}[x][[s]]$.

15. The smooth affine curve $C\colon x^3 + y^3 + 1$ is rigid ($T^1_C = H^1(C, \Theta_C) = 0$), so $C \to 0$ is an algebraic, formally versal object, yet the 1-parameter family $x^3 + y^3 + 1 + \lambda xy$ is non trivial. Show that a formal change of coordinates trivialises the family (hint: first consider the first order case). Why is it not convergent?

16. Compute T^1 for all A–D–E singularities.

17. Find the miniversal deformation of A_2. Describe the discriminant, i.e. the locus in the base space over which the fibres are singular. What type of singularities can be found in these fibres?

18. Let $X = \operatorname{Spec}(k[x_1, x_2, x_3, x_4]/(x_1, x_2, x_3, x_4)^2$ be the fat point of multiplicity 5. Compute T^1_X. (More work, but possible: compute T^2_X).

19. Let X be the union of the (x,y)-plane and the z-axis in $\mathbb{C}^3$. Compute the first order embedded deformations N_X in $\mathbb{C}^3$ and show that all are in the image of $\Theta_{\mathbb{C}^3} \otimes \mathscr{O}_X$ (i.e. $T^1_X = 0$). Interprete this geometrically.

20. Let X be the cone over the rational normal curve of degree 4, which is given by the 2×2 minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

a) Compute T^1_X.

b) If $\dim T^1_X = \tau$ choose parameters $\varepsilon_1, \dots, \varepsilon_\tau$ and write down a deformation of X over

$$\operatorname{Spec} k[\varepsilon_1, \dots, \varepsilon_\tau]/(\varepsilon_1, \dots, \varepsilon_\tau)^2$$

which is versal to first order

c) Try to lift your deformation from b) to a deformation over

$$\operatorname{Spec} k[\varepsilon_1, \dots, \varepsilon_\tau]/(\varepsilon_1, \dots, \varepsilon_\tau)^3 .$$

21. Let X consist of the coordinate axes in $\mathbb{C}^n$; equations are $z_i z_j = 0$ for $i \neq j$. Compute T^1_X.

22. Let $X = C_1 \cup C_2$ be the union of two transversally intersecting curves of genus g_1, $g_2 \geq 2$.

a) compute $p_a(X)$. Make an educated guess for the dimension of $\mathbb{T}^1_X$.

b) compute $H^1(X, \mathscr{T}^0)$. Hint: try to compute on the normalisation $\widetilde{X} = C_1 \coprod C_2$ and show that $H^1(X, \mathscr{T}^0) = H^1(C_1, \Theta_{C_1}(-P)) \oplus H^1(C_2, \Theta_{C_2}(-P))$ where $P = C_1 \cap C_2$ is the intersection point.

c) Compute the dimension of $\mathbb{T}^1_X$.

d) What happens if the genus is zero or one?

23. Let $0 \neq f \in k[x_1, \dots, x_n]$ and set $Y = V(f)$. Let $X = \mathbb{A}^1_k$ be a line and let $\varphi \colon X \to Y$ be any map. "Compute" the modules $T^0_{X/Y}$, $T^1_{X/Y}$ and $T^2_{X/Y}$.

24. Let $C \subset \mathbb{P}^2$ be a smooth octic in the plane given by a homogeneous polynomial $f_8(X, Y, Z)$ and consider the 'double octic' X obtained as two-fold covering of $\mathbb{P}^2$ branched along C, so X is given by $W^2 = f_8(X, Y, Z)$.

a) What is the Euler number of C? Use this to compute the Euler number of X.

b) On how many parameters does the construction of X depend (this is the as the number of parameters for C).

c) Use the adjunction formula to show that $K_X^2 = 2$. Recall that $K_X = \pi^* K_{\mathbb{P}^2} + B$ where $\pi_* B = C$.

d) Plug in the formula for $\chi(\Theta_X)$. What do you get? Does this fit with b)?

25. Compute the module $\Theta_X = \operatorname{Der}_k(\mathscr{O}_X, sierX)$ for the A_k-singularity $X = \{xy - z^{k+1} = 0\}$.

What derivation do you always n have on a weighted homogeneous hypersurface? Try to prove that there are no more. (A function $f \in k[x_0, \dots, k_n]$ is weighted homogeneous iff there exist positive numbers a_0, dots, a_n such that $f(t^{a_0} x_0, \dots, t^{a_n} x_n) = t f(x_0, \dots, x_n)$.)

26. Consider the quadric cone $XY - Z^2$ in $\mathbb{P}^3$. Blow up the vertex $(0 : 0 : 0 : 1)$ of the cone in $\mathbb{P}^3$ and show that the strict transform of the cone is the surface F_2.

27. Describe the two rulings on the standard model $XY - ZW = 0$ of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$. Consider the same equation in $k[x, y, z, w]$. Show that it defines a 3-dimensional A_1-singularity $X \subset \mathbb{A}^4$. Using a ruling one gets a codimension two subvariety $\widetilde{X}$ of $\mathbb{P}^1 \times \mathbb{A}^4$ given by the minors of

$$\begin{pmatrix} \alpha & x & w \\ \beta & z & y \end{pmatrix}$$

where $(\alpha : \beta)$ are coordinates on $\mathbb{P}^1$. Compute the fibres of the map $\widetilde{X} \to X$. Show that $\widetilde{X}$ is smooth. Hint: look at affine charts on $\mathbb{P}^1$.

28. Consider the deformation $XY - Z^2 + sW^2$ of the quadric cone. Show that for $s \neq 0$ the fibre is a smooth quadric isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Show that after a base change $s := t^2$ the total space has one singular point which can be resolved as in the previous exercise. Show that one obtains in this way a deformation of F_2 into $\mathbb{P}^1 \times \mathbb{P}^1$.

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Ideas

Here the main ideas we (Jan and Duco) have on the book (December 10, 1997).

(1) The photograph of the participants of the summer school in Nordfjordeid should be in the book! The book as it now represents fairly precise what we did then. It seems in retrospect that we did rather well, and we should not make very big structural changes. We suggest that we all should read and think about this first version of the book, discuss with each other (e-mail), and make corresponding changes later. Also, it seems important that we do this quickly, for obvious reasons.

(2) The book seems to consist globally of five blocks, application oriented ones seperated by more abstract ones. We think this was a good idea, and should be retained more or less. Additions and subtractions should be made with this structure in mind.

(3) We should include some clear proofs, in fact as much as possible, and especially if it is about deformations. E.g. the book should contain proofs that Plückers und Kleins ideas work. Or curves that really move in a surface. Or that the ADE really have simultaneous resolution and realise the Weyl-group quotient. But standard tools from alg. geometry only need good reference.

(4) It seems a good idea to collect references to the literature and add comments, historical remarks, credits, exercises at the end of each chapter.

(5) The book should contain more examples of miniversal but not universal families. One could think of more detailed descriptions deformation of A_1 and A_2 singularity. Examples of induced families. Schlessingers condition H4.

(6) Kodaira-Spencer map has to be covered better. By way of examples, but also its place in the general formalism. It occurs in the lectures only in the form of characteristic map. And of course in the isomorphism Tangent space to moduli=$H^1(\Theta)$.

(7) Formal smoothness should be explained more extensively.

(8) We should maybe include some clear examples or exercises of Functors NOT satisfying Schlessinger. (Are there any?) Convergence problems should be discussed. (What are these Dufour things really about?).

(9) There are several interesting topics we could add.

- Maps to singular spaces to give easily accessable examples of obstructions. And it has close relation to formal smoothness. Put in Block III?
- Deformations of algebras. More in the beginning, also good example of deforming something. Do not need flatness. You see cohomolocal things coming into the picture.
- Kodaira's theory of deforming surfaces with ordinary singularities. This fits very well in the spirit of the book. Add more to the end.

- The Calabi-Yau stuff should be extended a little bit, but not too much.
- We should not put in more, because otherwise it will become a never ending story.

(10) We have the following suggestions for more drastic changes.

- In chapter 2 the computation of $H^1(\Theta)$ should be removed. Instead, the counting of parameters by geometrical methods should be extended and discussed more completely. It would also be a good place to *define* the general notion of family of smooth complex spaces as submersion to parameter space and get the topological/differentiable triviality of the family in the forground. Change in complex structure only. Maybe the example F_2 and F_0. Jump phenomena. No moduli space?
- The part on the ADE singularities should be moved before the surfaces. We could extend on it, define resolutions, rational singularities in general. These things are used everywhere.

boek.sty for Lectures on Deformation Theory

Fonts

The file provides first of all abbreviations for fonts. The BlackBoardBold font

$$\mathbb{ABCDEFGHIJKLMNPQRTUVWXYZ}$$

are obtained by typing `\A` etc. Note that `\O` and `\S` are missing so $\mathbb{O}$ can only be obtained with `\mathbb O` and respectively for $\mathbb{S}$. Alternative names are `\integer` for $\mathbb{Z}$, `\proj` for $\mathbb{P}$, `\complex` for $\mathbb{C}$ and `\real` for $\mathbb{R}$.

These fonts are used for the cotangent complex $\mathbb{L}$ and for hypercohomology. This can be changed later maybe into $\mathbf{H}$, so for $\mathbb{F}$, $\mathbb{L}$, $\mathbb{T}$ and $\mathbb{H}$ the alternatives `\FF`, `\LL`, `\TT` and `\HH` are provided.

For the preferred script font for the book:

$$\mathscr{A\,B\,C\,D\,E\,F\,G\,H\,I\,J\,K\,L\,M\,N\,O\,P\,Q\,R\,S\,T\,U\,V\,W\,X\,Y\,Z}$$

one has to type `\sA` etc. The commands `\cO`, `\cI`, `\cL`, `\cR`, `\cN`, `\cT` and `\cX` have the same effect and are provided for compatibility with earlier notes.

Fraktur letters are `\gm` or `\mi` for $\mathfrak{m}$, `\gp` for $\mathfrak{p}$ and `\gq` for $\mathfrak{q}$.

The greek letters `\varepsilon` (ε) and `\varphi` (φ) have the abbreviations `\eps` and `\vp`. Note that `\epsilon` and `\phi` give ϵ and ϕ.

Furthermore there are abbreviations

`\uf`	$\underline{\mathbf{F}}$
`\uc`	$\underline{\mathbf{C}}$
`\ADE`	A–D–E
`\half`	$\frac{1}{2}$
`\tX` or `wtx`	$\widetilde{X}$

Arrows

`\inj`	$\hookrightarrow$
`\surj`	$\twoheadrightarrow$
`\linj`	$\lhook\joinrel\longrightarrow$
`\lsurj`	$\relbar\joinrel\twoheadrightarrow$
`\lra`	$\longrightarrow$
`\lla`	$\longleftarrow$
`\lma`	$\longmapsto$
`\implies`	$\Rightarrow$

A map $f\colon X \to Y$ (`$f\colon X\to Y$`, note the difference with `$f: X\to Y$`: $f : X \to Y$ just as `\mid` gives the same as `|` but with different spacing, correct for use in definitions of sets) can be written as $X \xrightarrow{f} Y$ by `$X \mapright{f} Y$`. Likewise $X \xrightarrow{\quad f \quad} Y$ by `$X \maplongright{f} Y$` and $X \xleftarrow{f} Y$ by `$X \mapleft{f} Y$`. We have

$$\Big\uparrow{\scriptstyle f} \qquad \Big\downarrow{\scriptstyle f}$$

by `\mapup{f}` and `\mapdown{f}`.

log like operators

`\Spec`	Spec	`\depth`	depth
`\Der`	Der	`\Ker`	Ker
`\Def`	Def	`\Im`	Im
`\Hom`	Hom	`\Coker`	Coker
`\Ext`	Ext	`\ob`	ob
`\Aut`	Aut	`\pd`	pd
`\ord`	ord	`\codim`	codim
`\supp`	supp		

Theorems

We have the following environments which have the syntax

```
\begin{theorem}
This is the text of a theorem.
\end{theorem}

\begin{defn}
This is the text of a definition.
\end{defn}

\begin{remark}
This is the text of a remark.
\end{remark}

\begin{proof}
This is the text of a proof.
\end{proof}
```

and have the effect:

Theorem 26.1. *This is the text of a theorem.*

Definition 26.2. This is the text of a definition.

Remark 26.3. This is the text of a remark.

Proof. This is the text of a proof. □

In the same style as **Theorem** we have: `proposition`, `corollary`, `algorithm`, `claim`. In the same style as Definition: `example`, `problem`, `comment`, `exe`, where the last one gives 'Exercise'.

In the same style as **Theorem** there is an environment where the name has to be specified:

Joke 26.4. *This is not funny.*

was obtained by typing:

```
\thm{Joke} This is not funny.
\endthm
```

Note that one should end with `\endthm` without braces!

An unnumbered **Definition** with name to be given:

Observation. Text, but no number.

is obtained by

```
\rmk{Observation} Text, but no number.
\endrmk
```

Centered headings

These are obtained with

```
\tussenkop{Centered headings}
```

Maybe these subsections should also be numbered.

Pictures

Picture files of type `.eps` are included with

`\plaatje[`*xcm*`]{`*file-name*`}`

where *xcm* is the optional `\epsfxsize` .